

Supplement to “Optimal Allocation with Peer Information”

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[Section 1](#) presents examples of the feasibility graph and stochastic extreme DIC mechanisms. [Section 2](#) discusses jury mechanisms.

1 Examples of the Feasibility Graph and Stochastic Extreme Points

1.1 The Feasibility Graph

[Figure 1](#) shows the feasibility graph in an example with three agents, where agents 1 and 2 each have two possible types, and agent 3 has three possible types. The left panel shows the set of type profiles, here depicted as a $2 \times 2 \times 3$ grid. Agent 1’s type varies horizontally, agent 2’s type vertically, and agent 3’s type diagonally. The right panel shows the feasibility graph. To understand the connection between the panels, think of a vertex (i, θ_{-i}) of the feasibility graph as a set of type profiles along which the types of agents other than i are fixed at θ_{-i} while i ’s type varies across all of Θ_i . In the left panel, for example, the vertex $(1, \theta_{-1})$ is thus depicted as the line connecting the profiles $\theta = (\theta_1, \theta_2, \theta_3)$ and $(\theta'_1, \theta_2, \theta_3)$. Two vertices are adjacent if and only if the corresponding sets of type profiles intersect; for example, the vertices $(1, \theta_{-1})$, $(2, \theta_{-2})$ and $(3, \theta_{-3})$ are all adjacent since the corresponding sets of type profiles intersect at θ . In the right panel, vertices are depicted as circles (instead of lines) and lines (both solid and dashed ones) indicate adjacencies.

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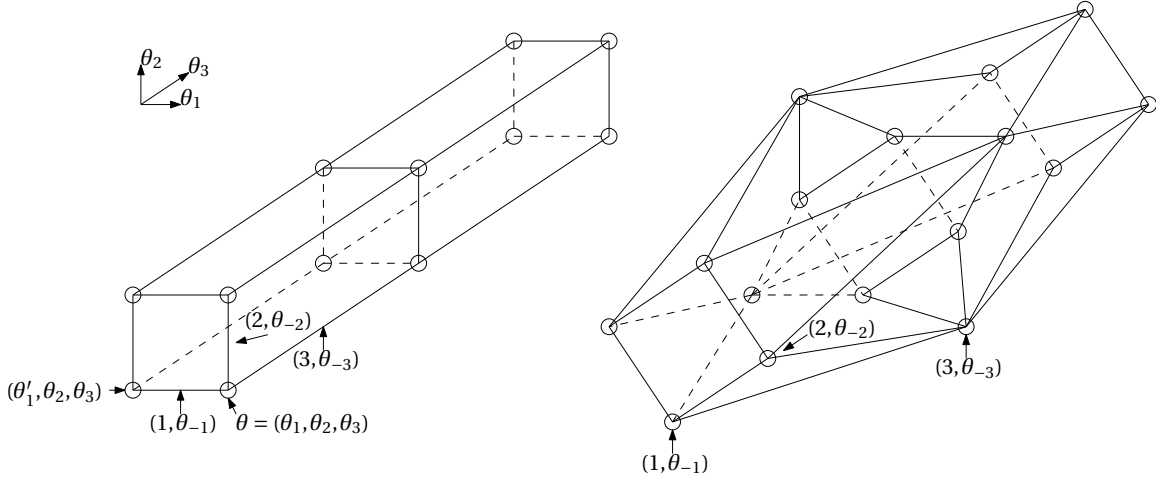


Figure 1: The set of type profiles (left) and the feasibility graph (right) in an example with $n = 3$, $|\Theta_1| = |\Theta_2| = 2$ and $|\Theta_3| = 3$.

1.2 A Stochastic Extreme Point

Continuing with the example from [Section 1.1](#), suppose there are three agents, where agents 1 and 2 each have two possible types, and agent 3 has three possible types. The left panel of [Figure 2](#) shows the space of type profiles. Agent 1's type varies horizontally, agent 2's type vertically, and agent 3's type diagonally.

The right panel of [Figure 2](#) shows an odd hole $v_1, \dots, v_7$ in the feasibility graph. Let us say that a profile θ *contains* the vertices $\{(i, \theta_{-i}) : i \in [n]\}$. For each $\ell \in \{1, \dots, 7\}$, the type profile θ^ℓ indicated in the left panel of the figure contains the vertices $v_{\ell-1}$ and v_ℓ (where $v_0 = v_7$ is understood). For example, type profile θ^1 contains v_7 and v_1 , and θ^2 contains v_1 and v_2 , and so on.

Our candidate stochastic extreme DIC mechanism q assigns $1/2$ to all vertices $v_1, \dots, v_7$ of the odd hole, and 0 to all other vertices. Translated to the left panel, this means that, for each $\ell \in \{1, \dots, 7\}$, at θ^ℓ the mechanism flips a coin between the two agents who have vertices from the hole which are contained in θ^ℓ . For example, at θ^1 the mechanism flips a coin between agents 1 and 3, at θ^2 between agents 3 and 2, and so on.

In the main text, we intuited that randomization helps resolve the trade-off between allocating to an agent and using the agent's information since the agent can simultaneously win with some probability but nevertheless influence how the remaining probability is distributed among the other agents. We see this in the

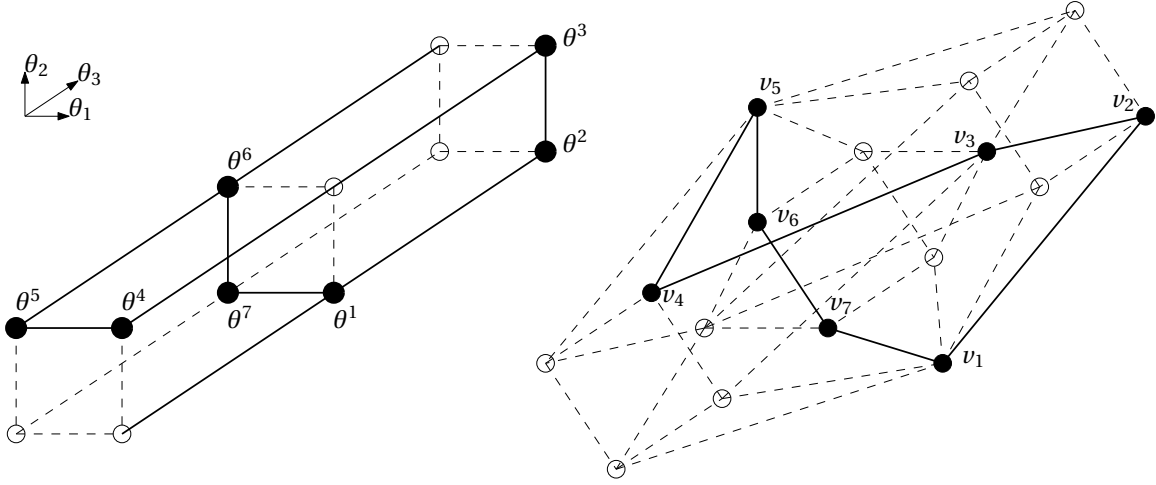


Figure 2: The right panel shows an odd hole $v_1, \dots, v_7$ in the feasibility graph. The left panel shows the seven type profiles $\theta^1, \dots, \theta^7$ that each contain two vertices of the hole.

stochastic mechanism q . For example, agent 3 wins with probability $1/2$ at both θ^1 and θ^2 . These two type profiles differ only in agent 3's type. Depending on agent 3's report, the remaining probability $1/2$ is assigned to agent 1 (at θ^1) or to agent 2 (at θ^2). In this sense, the principal enjoys both the benefit from allocating to agent 3 and agent 3's information.

To sharpen the intuition, suppose the weights $w_i(\theta_{-i}) = \mu(\theta_{-i})u_i(\theta_{-i})$ are such that $w_i(\theta_{-i})$ equals 1 if the vertex (i, θ_{-i}) is in the odd hole $\{v_1, \dots, v_7\}$, and equals -1 otherwise. In particular, at θ^1 the principal wishes to select agent 1 or 3, while at θ^2 the principal wishes to select agent 2 or 3. Thus, here it is valuable to allocate to agent 3, and agent 3's type also contains valuable information.

Let us show that for these weights the stochastic DIC mechanism q described earlier is in fact uniquely optimal (and therefore extreme). The principal's expected utility from q is given by $\sum_v w(v)q(v) = 7/2$. Now let q' be an optimal DIC mechanism. We show $q = q'$. Since q' is optimal, it holds $\sum_v w(v)q'(v) \geq \sum_v w(v)q(v) = 7/2$. For vertices outside $\{v_1, \dots, v_7\}$, the weight w is strictly negative. Since decreasing the allocation probability at a vertex only slackens feasibility, we must have $q'(v) = 0$ for all v outside $\{v_1, \dots, v_7\}$. Thus $\sum_v w(v)q'(v) = \sum_{\ell=1}^7 q'(v_\ell)$ and it suffices to show $q'(v_\ell) = 1/2$ for all $\ell \in [7]$. For all $\ell \in [7]$, feasibility requires $q'(v_{\ell-1}) + q'(v_\ell) \leq 1$ since v_ℓ and $v_{\ell-1}$ are adjacent (where again $v_0 = v_7$). Therefore, for all $\ell \in [7]$, the inequality $7/2 \leq q'(v_1) + \dots + q'(v_7)$ requires $7/2 \leq q'(v_\ell) + 3$, and thus $1/2 \leq q'(v_\ell)$.

Using again $q'(v_{\ell-1}) + q'(v_{\ell}) \leq 1$, we conclude $q(v_{\ell}) = 1/2$ for all $\ell \in [7]$, as desired.

2 Jury mechanisms

In this appendix, we discuss jury mechanisms. Jury mechanisms resemble peer review procedures used in practice. Therefore, it is interesting to understand when jury mechanisms perform well.

Definition 1 (Jury mechanisms). A mechanism q is a *jury mechanism* if there is a partition of the set of agents into *jurors* J and *candidates* C such that:

- (1) jurors are never allocated the good ($q_i = 0$ for all $i \in J$);
- (2) candidates' reports never influence the allocation (q is constant in θ_i for all $i \in C$).

Note, constant mechanisms are special cases of jury mechanisms (let all agents be candidates). Agents who do not influence the allocation and are never allocated the good may be defined either as jurors or candidates.

In [Section 2.1](#) we identify the special cases of the model in which jury mechanisms are exactly optimal among DIC mechanisms. In [Section 2.2](#) we discuss when jury mechanisms are approximately optimal with many agents.

2.1 Exact Optimality

The next result characterizes the special cases in which jury mechanisms are optimal.

Theorem 1. *If $n = 2$, then every DIC mechanism is a randomization over jury mechanisms; in particular, a jury mechanism is optimal.*

If $n \leq 3$, then every DIC mechanism q that always allocates the good (i.e., $\sum_{i \in [n]} q_i = 1$) is a randomization over jury mechanisms; in particular, a jury mechanism is optimal among mechanisms that always allocate.

The proof follows in [Section 2.3](#).

With two agents, a jury mechanism involves one juror who recommends (based only on this juror's information) whether the other agent (the candidate) should be allocated the good or whether the principal should keep the good. With three agents and mandatory allocation, there is one juror who recommends (based only on this

juror’s information) which of the two remaining agents (the candidates) should be allocated the good. The nomination of some agent as the juror trades-off, on the one hand, this agent’s peer information regarding the other agent(s), and, on the other hand, the possibility of allocating to this agent if another agent were the juror.

Our results on stochastic extreme points and complexity imply that jury mechanisms are not generally optimal outside the special cases identified by [Theorem 1](#). The same is true for DIC mechanisms with mandatory allocation; see the working paper version (Niemeyer and Preusser (2025)). Thus, [Theorem 1](#) identifies the maximal set of environments in which jury mechanisms are optimal for all information structures μ .

[Theorem 1](#) generalizes results by Holzman and Moulin (2013, Proposition 2.i) and Kato and Ohseto (2004, Theorem 5, obtained in the context of exchange economies). Their results can be interpreted in our context as showing that all three-agent *deterministic* DIC mechanisms that always allocate the good are jury mechanisms. [Theorem 1](#) extends this characterization to stochastic mechanisms.

2.2 Approximate Optimality

First, we show that vanishing informational size does not imply that jury mechanisms are approximately optimal with many agents. Then, we give notions of exchangeability under which jury mechanisms are approximately optimal with many agents.

Vanishing Informational Size is Insufficient for Approximate Optimality.

Here, we show that jury mechanisms need not be approximately optimal along regular sequences of environments with vanishing informational size. The basic building block is an environment with three agents that we later expand to have many agents. The key properties of the environment are as follows:

- (1) At each type profile in the support of the type distribution, exactly two agents have a peer value of 1, while the other agent has a peer value of 0. Further, an agent’s peer value is 0 if the reports of the others is outside the support.
- (2) The information of an individual agent correctly identifies only with probability $2/3$ which of the other two agents has a peer value of 1 (possibly both).
- (3) Each individual agent has a peer value of 1 at only $2/3$ of all type profiles in the support of the type distribution.

In such an environment with three agents, the argument for why ranking-based

mechanisms outperform jury mechanisms is as follows. Informational size is at most $2/3$ at every profile since agents influence the ranks only through the tie-breaking rule. Property 1 implies that the ranking-based mechanism with threshold $p = 2/3$ always selects an agent with peer value 1; indeed, an agent with peer value 1 has rank $2/3$ or less regardless of what this agent reports, while an agent with peer value 0 has rank 1; thus at each profile the two agents with peer value 1 both enjoy a winning probability $1/(np) = 1/2$. However, properties 2 and 3 imply that each jury mechanism obtains at most $2/3$. Indeed, if there is a single juror, then property (2) implies the juror finds the best candidate only with probability $2/3$. For fewer or more jurors, the jury mechanism is constant, and hence property (3) implies that the constantly chosen candidate is the best only with probability $2/3$.

An explicit environment with the above three properties is as follows. For each of the three agents, the type space is $\{1, 2, 3\}$. Consider the following odd 9-hole H in the feasibility graph:

$$(\cdot, 1, 1), (2, \cdot, 1), (2, 2, \cdot), (\cdot, 2, 2), (3, \cdot, 2), (3, 3, \cdot), (\cdot, 3, 3), (1, \cdot, 3), (1, 1, \cdot).$$

Let T denote the set of type profiles traversed by this odd hole; that is,

$$T = \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (2, 2, 2), (3, 2, 2), (3, 3, 2), (3, 3, 3), (1, 3, 3), (1, 1, 3)\}.$$

The distribution of type profiles is uniform over T (i.e., $\mu(\theta) = (1/9)\mathbf{1}(\theta \in T)$ for all $\theta \in \Theta$) and peer values are given by $\bar{u}_i(\theta_{-i}) = \mathbf{1}((i, \theta_{-i}) \in H)$. Properties (1) and (3) are immediate from the construction. For property (2), it suffices to note that for every distinct agents i and j and type θ_i , agent j has a peer value of 1 at no more than two of the three type profiles where i 's type is θ_i ; for example, for $i = 3$ and $\theta_3 = 1$, agent 1 has a peer value of 1 only at $(1, 1, 1)$ and $(2, 1, 1)$ (but not at $(2, 2, 1)$), while agent 2 has a peer value of 1 only at $(2, 1, 1)$ and $(2, 2, 1)$ (but not at $(1, 1, 1)$).

We now expand the above example to have many agents and so that informational size vanishes as the number of agents diverges and regularity holds. Let the number of agents n be an integer multiple ℓ of 3. Partition the agents into ℓ groups of 3 so that within each group the agents are labeled with three consecutive integers (the labeling only matters for the tie-breaking rule). For each group, fix an auxiliary environment for three agents as above. Define the environment (for n agents) by letting all variables from the auxiliary environments be independent across groups. (Thus, for

example, Nature takes ℓ independent uniform draws from the set T defined above.) Informational size is at most $2/n$ at every type profile in the support since each agent cannot influence the peer values of the agents belonging to other groups, and since within each group each agent influences the rank only through the tie-breaking rule.¹ Further, as $n \rightarrow \infty$ the sequence of environments is regular since two thirds of the agents all have the highest peer value of 1. As before, a ranking-based mechanism (with, say, threshold $2/3$) always selects an agent with the highest peer value. Now consider a jury mechanism. Fix a realization of the types of the jurors. For every group ℓ , the jurors find a candidate from group ℓ whose peer value is 1 only with probability $2/3$; this is because the types of jurors outside of group ℓ are uninformative about the candidates in group ℓ , and because the jurors from group ℓ find a candidate whose peer value is 1 only with probability $2/3$. There is only one good to allocate. Thus, regardless from which group the principal selects a candidate using the types of the jurors, the principal finds a candidate whose peer value is 1 only with probability $2/3$. Thus, the expected utility of each jury mechanism is at most $2/3$.

Sufficient Conditions for Approximate Optimality. Here, we formalize two notions of exchangeability under which jury mechanisms are approximately optimal with many agents. We focus on environments in which agents receive conditionally independent signals about the values of each other.

An *information structure with conditionally independent signals* specifies the following for every $i \in \mathbb{N}$:

- (1) a distribution f_i whose support $\text{supp } f_i$ is finite and contained in $[-1, 1]$;
- (2) for every $j \in \mathbb{N}$ distinct from i a finite set S_i^j and a function g_i^j that assigns to each $u_i \in \text{supp } f_i$ a distribution over S_i^j .

Given an information structure with conditionally independent signals and $n \in \mathbb{N}$, the environment with n agents works as follows. The values $u_1, \dots, u_n$ are independent across agents 1 to n with respective marginals $f_1, \dots, f_n$. Conditional on u_i , each agent j other than i observes a signal σ_i^j about u_i drawn from $g_i^j(\cdot | u_i)$. Thus, each agent j 's type is $(\sigma_i^j)_{i \in [n]: i \neq j}$. Conditional on $(u_1, \dots, u_n)$, the signals $(\sigma_i^j)_{i \neq j}$ are independent. Thus, the probability of a profile $((u_i)_{i \in [n]}, (\sigma_i^j)_{i, j \in [n]: i \neq j})$ is $\prod_{i, j \in [n]: i \neq j} f_i(u_i) g_i^j(\sigma_i^j | u_i)$.

Nothing in the analysis would change if an agent i also observed their own value

¹For this argument, the partition of the agents into groups should be chosen so that the agents within a group are labeled with three consecutive integers. Since ties are broken in favor of agent i versus agent j if and only if $i < j$, each agent can influence their own rank by at most $2/n$.

since this information cannot be used to determine agent i 's allocation (by DIC) and since agent i 's value is uninformative about the other agents' values (by independence of the values).

Fix an information structure with conditionally independent signals.

- *Agents are exchangeable as suppliers of information* if for all $i \in \mathbb{N}$ the set of possible signals S_i^j and the conditional distributions $(g_i^j(\cdot|u_i))_{u_i \in \text{supp } f_i}$ do not depend on j .

Intuitively, for all i and j , the information that j can potentially provide about i does not depend systematically on j 's identity.

- *Agents are exchangeable as recipients of the good* if the marginal distributions f_i do not depend on i and for all $j \in \mathbb{N}$ the set of possible signals S_i^j and all conditional distributions $(g_i^j(\cdot|u))_{u \in \text{supp } f_i}$ do not depend on i .

Intuitively, for all i and j , neither the value of allocating to i nor the information that j can potentially provide about i depend systematically on i 's identity.

Theorem 2. *Consider an information structure with conditionally independent signals in which agents are exchangeable suppliers of information or in which agents are exchangeable as recipients of the good. As the number of agent diverges, the difference between the principal's expected utility from an optimal jury mechanism and an optimal DIC mechanism vanishes.*

We provide a proof sketch that is straightforward but notationally tedious to verify. For this proof, it is crucial that the information structure is fixed. (By contrast, our result for ranking-based mechanisms, Theorem 3 in the main text, does not require a fixed information structure.)

Suppose agents are exchangeable as suppliers of information. Let $n \in \mathbb{N}$. In the environment with n agents, let $\bar{U}^n$ denote the expected utility from always allocating to an agent with the highest peer value. Each DIC mechanism in the environment with n agents obtains at most $\bar{U}^n$. We show that $\bar{U}^n$ is attainable in the environment with $2n$ agents using a jury mechanism. The sequence $(\bar{U}^n)_{n \in \mathbb{N}}$ converges,² and hence the theorem follows. For all n , the value $\bar{U}^n$ is generated by the following mechanism q^n (which need not be DIC): the principal collects the entire profile of signals $(\sigma_i^j)_{i,j \in [n]: i \neq j}$, and then allocates to an agent i for which the expected value of u_i conditional on

²The sequence converges since it is bounded and increasing. Boundedness is immediate since values are in $[-1, 1]$. Monotonicity follows since the information structure is held fixed as the number of agents increases; the principal can ignore agents who are added to the environment.

$(\sigma_i^j)_{j \in [n]: i \neq j}$ is maximal across $i \in [n]$. We shall replicate q^n using the following jury mechanism with $2n$ agents: agents 1 to n are the candidates, agents $n + 1$ to $2n$ are the jurors. For each $j \in [n]$, the principal collects from juror $n + j$ only the signals $(\sigma_i^{n+j})_{i \in [n]: i \neq j}$. The principal treats the signals thus obtained in the same way that q^n would treat the signals of agent j . That is, the principal selects a candidate i with the highest expected value u_i conditional on $(\sigma_i^{n+j})_{j \in [n]: j \neq i}$. The only difference between this jury mechanism and q^n is that for each $i \in [n]$ the conditional expected value of agent i is computed using $n - 1$ signals originating from agents $n + 1$ to $2n$ instead of $n - 1$ signals originating from $[n] \setminus \{i\}$. But agents are exchangeable as suppliers of information, meaning that the signals about u_i are conditionally iid. across *all* agents other than i . Thus, the origin of the signals is irrelevant for determining the conditional expected value of u_i . Thus, the described jury mechanism yields the same expected utility as q^n .

If agents are exchangeable as recipients of the good, a similar argument applies. We now construct a jury mechanism in which agents $n + 1$ to $2n$ substitute for agents 1 to n whenever one of the latter is selected to receive the good by q^n . That is, agents 1 to n are the jurors, agents $n + 1$ to $2n$ are the candidates. For all $j \in [n]$, the principal collects from juror j only the signals $(\sigma_i^j)_{i \in \{n+1, \dots, 2n\}: i \neq n+j}$.³ The principal treats the signal of $j \in \{1, \dots, n\}$ about $i \in \{n + 1, \dots, n\}$ in the same way that q^n would treat the signal of j about agent $n - i$. Since agents are exchangeable as recipients, the conditional value of allocating to agent $i \in \{n + 1, \dots, n\}$ given the elicited signal profile equals the conditional value of allocating to agent $n - i$ if the same profile of signals had been reported about about $n - i$. Thus, the described jury mechanism achieves the same expected utility as q^n .

2.3 Proof of Theorem 1

If $n = 2$ and q is a DIC mechanism that always allocates, then $q_1(\theta_2) + q_1(\theta_1) = 1$ for all $(\theta_1, \theta_2) \in \Theta$. In particular, q is constant. A constant mechanism is a convex combination of constant deterministic mechanisms. All constant mechanisms are jury mechanisms.

For $n = 2$, a DIC mechanism (that need not always allocate) is a special DIC

³We exclude the signal if $i = n + j$ since q^n computes each agent's conditional expected value using $n - 1$ signals. Thus, to replicate q^n , we deliberately choose the jury mechanism to also use only $n - 1$ signals for each conditional expected value.

mechanism with three agents that always allocates and where one agent is a default agent whose report does not influence the allocation but absorbs all residual allocation probability.

Thus, we prove the claim if $n = 3$ for DIC mechanisms that always allocate. In what follows, let $n = 3$, and let $\bar{Q}$ denote set of DIC mechanisms that always allocate.

In our terminology, Holzman and Moulin (2013, Proposition 2.ii) show that all deterministic mechanisms in $\bar{Q}$ are jury mechanisms. Hence, to prove that all mechanisms in $\bar{Q}$ are convex combinations of jury mechanisms, it suffices to show that all extreme points of $\bar{Q}$ are deterministic. Thus, let $q \in \bar{Q}$ be stochastic. We show q is not an extreme point of $\bar{Q}$.

Let $\tilde{V}$ denote the subset of vertices v of the feasibility graph such that $q(v) \in (0, 1)$. We construct non-empty disjoint subsets R (red) and B (blue) of $\tilde{V}$ such that, for all maximal cliques X , either $X \cap (R \cup B) = \emptyset$ or $|X \cap R| = |X \cap B| = 1$. Recall that we may identify maximal cliques with type profiles. If $X \cap (R \cup B) = \emptyset$, we say that the type profile (associated with X) is *uncolored*; if $|X \cap R| = |X \cap B| = 1$, we say that the type profile is *two-colored*.

The existence of such sets R and B imply that q is not an extreme point. Indeed, for a sufficiently small number $\varepsilon > 0$, both $q + \varepsilon(\mathbf{1}_R - \mathbf{1}_B)$ and $q - \varepsilon(\mathbf{1}_R - \mathbf{1}_B)$ are in $\bar{Q}$, and q is a convex combination of $q + \varepsilon(\mathbf{1}_R - \mathbf{1}_B)$ and $q - \varepsilon(\mathbf{1}_R - \mathbf{1}_B)$. Here, $\mathbf{1}_R$ and $\mathbf{1}_B$, respectively, denote the indicator functions for R and B , respectively.

We assume without loss that there exists a type profile $(\theta_1^0, \theta_2^0, \theta_3^0)$ such that the vertices $(\cdot, \theta_2^0, \theta_3^0)$ and $(\theta_1^0, \cdot, \theta_3^0)$ are in $\tilde{V}$. Define $T_1 = \{\theta_1 \in \Theta_1 : (\theta_1, \cdot, \theta_3^0) \notin \tilde{V}\}$ and $T_2 = \{\theta_2 \in \Theta_2 : (\cdot, \theta_2, \theta_3^0) \notin \tilde{V}\}$. Let $T_1^c = \Theta_1 \setminus T_1$ and $T_2^c = \Theta_2 \setminus T_2$. Both T_1^c and T_2^c are non-empty since $(\cdot, \theta_2^0, \theta_3^0)$ and $(\theta_1^0, \cdot, \theta_3^0)$ are in $\tilde{V}$.

First, if T_1 and T_2 are both empty, then the sets $R = \{(\theta_1, \cdot, \theta_3^0) : \theta_1 \in \Theta_1\}$ and $B = \{(\cdot, \theta_2, \theta_3^0) : \theta_2 \in \Theta_2\}$ have the desired properties.

Henceforth, let $T_1 \neq \emptyset$ (the case $T_2 \neq \emptyset$ being analogous). We distinguish two cases.

Case 1. Let $T_2 \neq \emptyset$. Define $\tilde{\Theta}_3$ as the set of $\theta_3 \in \Theta_3$ such that for all $\theta_1 \in T_1^c$ and $\theta_2 \in T_2^c$ we have $(\theta_1, \cdot, \theta_3) \in \tilde{V}$ and $(\cdot, \theta_2, \theta_3) \in \tilde{V}$. Note $\tilde{\Theta}_3$ is non-empty since $\theta_3^0 \in \tilde{\Theta}_3$. We make the following observations.

- **Observation 1.1.** If $(\theta_1, \theta_2) \in T_1 \times T_2$, then $(\theta_1, \theta_2, \cdot) \notin \tilde{V}$.

Indeed, if $(\theta_1, \theta_2, \cdot) \in \tilde{V}$, then only agent 3 enjoys an interior winning probability at the profile $(\theta_1, \theta_2, \theta_3^0)$, which is impossible since the object is always allocated.

- **Observation 1.2.** If $(\theta_1, \theta_2) \in (T_1 \times T_2^c) \cup (T_1^c \times T_2)$, then $(\theta_1, \theta_2, \cdot) \in \tilde{V}$.

Indeed, the assumption $(\theta_1, \theta_2) \in (T_1 \times T_2^c) \cup (T_1^c \times T_2)$ implies that exactly one of agent 1 and agent 2 enjoys an interior winning probability at the profile $(\theta_1, \theta_2, \theta_3^0)$; hence agent 3 must also enjoy an interior winning probability at this profile.

- **Observation 1.3.** *If $\theta_3 \notin \tilde{\Theta}_3$, then all $\theta_1 \in T_1$ and $\theta_2 \in T_2$ satisfy $(\theta_1, \cdot, \theta_3) \in \tilde{V}$ and $(\cdot, \theta_2, \theta_3) \in \tilde{V}$.*

Indeed, let $\theta_3 \notin \tilde{\Theta}_3$. By definition of $\tilde{\Theta}_3$, there exists $\theta'_1 \in T_1^c$ or $\theta'_2 \in T_2^c$ such that $(\theta'_1, \cdot, \theta_3) \notin \tilde{V}$ or $(\cdot, \theta'_2, \theta_3) \notin \tilde{V}$. We first prove the claim in the case in which there is $\theta'_2 \in T_2^c$ such that $(\cdot, \theta'_2, \theta_3) \notin \tilde{V}$.

Let $\theta_1 \in T_1$. We show $(\theta_1, \cdot, \theta_3) \in \tilde{V}$. Since $(\theta_1, \theta'_2) \in T_1 \times T_2^c$, we have $(\theta_1, \theta'_2, \cdot) \in \tilde{V}$ (Observation 1.2). Since also $(\cdot, \theta'_2, \theta_3) \notin \tilde{V}$ (by assumption), feasibility at $(\theta_1, \theta'_2, \theta_3)$ requires $(\theta_1, \cdot, \theta_3) \in \tilde{V}$, as desired.

Now let $\theta_2 \in T_2$. We show $(\cdot, \theta_2, \theta_3) \in \tilde{V}$. By assumption, T_1 is non-empty. Find $\theta_1 \in T_1$. By the previous paragraph, $(\theta_1, \cdot, \theta_3) \in \tilde{V}$. Since $(\theta_1, \theta_2) \in T_1 \times T_2$, we know $(\theta_1, \theta_2, \cdot) \notin \tilde{V}$ (Observation 1.1). Thus, since $(\theta_1, \cdot, \theta_3) \in \tilde{V}$, feasibility at $(\theta_1, \theta_2, \theta_3)$ requires $(\cdot, \theta_2, \theta_3) \in \tilde{V}$, as desired.

We have proven the claim in the case in which there is $\theta'_2 \in T_2^c$ such that $(\cdot, \theta'_2, \theta_3) \notin \tilde{V}$. If there exists $\theta'_1 \in T_1^c$ such that $(\theta'_1, \cdot, \theta_3) \notin \tilde{V}$, the argument is analogous with the roles of agents 1 and 2 switched. Switching the roles is valid since the claims proven so far hold symmetrically for both agents 1 and 2 and since both T_1 and T_2 are non-empty.

We next define our candidates for R and B . We first define the following:

$$\begin{aligned} \forall \theta_3 \in \tilde{\Theta}_3, \quad R'(\theta_3) &= \{(\cdot, \theta_2, \theta_3) : \theta_2 \in T_2^c\} \quad \text{and} \quad B'(\theta_3) = \{(\theta_1, \cdot, \theta_3) : \theta_1 \in T_1^c\}; \\ \forall \theta_3 \notin \tilde{\Theta}_3, \quad R'(\theta_3) &= \{(\theta_1, \cdot, \theta_3) : \theta_1 \in T_1\} \quad \text{and} \quad B'(\theta_3) = \{(\cdot, \theta_2, \theta_3) : \theta_2 \in T_2\}. \end{aligned}$$

Let $R_0 = \{(\theta_1, \theta_2, \cdot) : \theta_1 \in T_1^c, \theta_2 \in T_2\}$ and $B_0 = \{(\theta_1, \theta_2, \cdot) : \theta_1 \in T_1, \theta_2 \in T_2^c\}$. Finally, let $R = R_0 \cup (\cup_{\theta_3 \in \tilde{\Theta}_3} R'(\theta_3))$ and $B = B_0 \cup (\cup_{\theta_3 \in \tilde{\Theta}_3} B'(\theta_3))$.

We next verify $R \cup B \subseteq \tilde{V}$. Observation 1.2 asserts $R_0 \cup B_0 \subseteq \tilde{V}$. Next, for all $\theta_3 \in \tilde{\Theta}_3$, the definition of $\tilde{\Theta}_3$ immediately implies $R'(\theta_3) \cup B'(\theta_3) \subseteq \tilde{V}$, and for all $\theta_3 \notin \tilde{\Theta}_3$ Observation 1.3 implies $R'(\theta_3) \cup B'(\theta_3) \subseteq \tilde{V}$. The sets R and B are disjoint and non-empty (since T_1^c , T_2^c and $\tilde{\Theta}_3$ are all non-empty).

To complete the proof in Case 1, it remains to verify all type profiles are two-colored or uncolored. By inspection, one may verify the following: If $\theta_3 \in \tilde{\Theta}_3$, then all type

profiles in $T_1 \times T_2 \times \{\theta_3\}$ are uncolored, whereas all other type profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3\}$ are two-colored. Conversely, if $\theta_3 \notin \hat{\Theta}_3$, then all type profiles in $T_1^c \times T_2^c \times \{\theta_3\}$ are uncolored, whereas all other type profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3\}$ are two-colored.

Case 2. Let $T_2 = \emptyset$. Thus, all $\theta_2 \in \Theta_2$ satisfy $(\cdot, \theta_2, \theta_3^0) \in \tilde{V}$. Define $\hat{\Theta}_3$ as the set of $\theta_3 \in \Theta_3$ such that all $\theta_2 \in \Theta_2$ satisfy $(\cdot, \theta_2, \theta_3) \in \tilde{V}$. Define $\hat{T}_1$ as the set of $\theta_1 \in \Theta_1$ for which there exists $\theta_3 \in \hat{\Theta}_3$ such that $(\theta_1, \cdot, \theta_3) \notin \tilde{V}$. Notice that $\hat{\Theta}_3$ is non-empty since $\theta_3^0 \in \hat{\Theta}_3$; the set $\hat{T}_1$ may or may not be empty. We observe the following:

- **Observation 2.1.** *If $(\theta_1, \theta_2) \in \hat{T}_1 \times \Theta_2$, then $(\theta_1, \theta_2, \cdot) \in \tilde{V}$.*

Indeed, by the definition of $\hat{T}_1$, there is $\theta_3 \in \hat{\Theta}_3$ such that $(\theta_1, \cdot, \theta_3) \notin \tilde{V}$. By definition of $\hat{\Theta}_3$, we have $(\cdot, \theta_2, \theta_3) \in \tilde{V}$. Hence feasibility at $(\theta_1, \theta_2, \theta_3)$ requires $(\theta_1, \theta_2, \cdot) \in \tilde{V}$.

- **Observation 2.2.** *If $\theta_3 \notin \hat{\Theta}_3$, then all $\theta_1 \in \hat{T}_1$ satisfy $(\theta_1, \cdot, \theta_3) \in \tilde{V}$.*

Indeed, by definition of $\hat{\Theta}_3$, there exists $\theta_2 \in \Theta_2$ such that $(\cdot, \theta_2, \theta_3) \notin \tilde{V}$. Observation 2.1 implies $(\theta_1, \theta_2, \cdot) \in \tilde{V}$. Hence feasibility at $(\theta_1, \theta_2, \theta_3)$ requires $(\theta_1, \cdot, \theta_3) \in \tilde{V}$.

We next define our candidates for R and B . We first define the following:

$$\begin{aligned} \forall \theta_3 \in \hat{\Theta}_3, \quad R'(\theta_3) &= \{(\cdot, \theta_2, \theta_3) : \theta_2 \in \Theta_2\}; \\ \forall \theta_3 \notin \hat{\Theta}_3, \quad R'(\theta_3) &= \{(\theta_1, \cdot, \theta_3) : \theta_1 \in \hat{T}_1\}; \\ R &= \cup_{\theta_3 \in \Theta_3} R'(\theta_3); \\ B &= \{(\theta_1, \cdot, \theta_3) : \theta_1 \notin \hat{T}_1, \theta_3 \in \hat{\Theta}_3\} \cup \{(\theta_1, \theta_2, \cdot) : \theta_1 \in \hat{T}_1, \theta_2 \in \Theta_2\}. \end{aligned}$$

Observations 2.1 and 2.2 and the definition of $\hat{\Theta}_3$ imply $R \cup \{(\theta_1, \theta_2, \cdot) : \theta_1 \in \hat{T}_1, \theta_2 \in \Theta_2\} \subseteq \tilde{V}$. The definition of $\hat{T}_1$ implies $\{(\theta_1, \cdot, \theta_3) : \theta_1 \notin \hat{T}_1, \theta_3 \in \hat{\Theta}_3\} \subseteq \tilde{V}$. The sets R and B are clearly disjoint, and they are non-empty since $\hat{\Theta}_3$ is non-empty.

To complete the proof in Case 2, it remains to show all type profiles are two-colored or uncolored. By inspection of R and B , one may verify the following: If $\theta_3 \in \hat{\Theta}_3$, then all type profiles in $\Theta_1 \times \Theta_2 \times \{\theta_3\}$ are two-colored; if $\theta_3 \notin \hat{\Theta}_3$, then all type profiles in $\hat{T}_1 \times \Theta_2 \times \{\theta_3\}$ are two-colored whereas all type profiles in $\hat{T}_1^c \times \Theta_2 \times \{\theta_3\}$ are uncolored. $\square$

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