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Dynamic games with (almost) perfect information

Wei He
Department of Economics, The Chinese University of Hong Kong

Yeneng Sun
Risk Management Institute, Departments of Economics and Mathematics, National University of Singapore

This paper aims to solve two fundamental problems on finite- or infinite-horizon dynamic games with complete information. Under some mild conditions, we prove the existence of subgame-perfect equilibria and the upper hemicontinuity of equilibrium payoffs in general dynamic games with simultaneous moves (i.e., almost perfect information), which go beyond previous works in the sense that stagewise public randomization and the continuity requirement on the state variables are not needed. For alternating move (i.e., perfect-information) dynamic games with uncertainty, we show the existence of *pure-strategy* subgame-perfect equilibria as well as the upper hemicontinuity of equilibrium payoffs, extending the earlier results on perfect-information deterministic dynamic games.

Keywords. Dynamic games, perfect information, almost perfect information, subgame-perfect equilibrium, atomless transition, atomless reference measure.

JEL classification. C62, C73.

1. Introduction

Dynamic games with complete information, where players observe the history and move simultaneously or alternately in a finite or infinite horizon, arise naturally in many situations. As noted in Chapter 4 of Fudenberg and Tirole (1991) and Harris et al. (1995), these games form a general class of regular dynamic games with many applications in economics, political science, and biology. The associated notion of subgame-perfect

Wei He: hewei@cuhk.edu.hk
Yeneng Sun: ynsun@nus.edu.sg

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equilibrium is a fundamental game-theoretic concept. For alternating move games with finite actions, an early result on backward induction (subgame-perfect equilibrium) was presented in Zermelo (1913). For simultaneous move games with finitely many actions and stages, the existence of subgame-perfect equilibria was shown in Selten (1965), while the infinite horizon but finite-action case was covered by Fudenberg and Levine (1983) under the usual continuity at infinity condition.[1]

Since the agents in many economic models need to make continuous choices, it is important to consider dynamic games with general action spaces. However, a simple example without any subgame-perfect equilibrium was presented in Harris et al. (1995), where the game has two players in each of the two stages with only one player having a continuous choice set.[2] Thus, the existence of subgame-perfect equilibria under some suitable conditions remains an open problem even for two-stage dynamic games. The purpose of this paper is to prove the existence of subgame-perfect equilibria for general dynamic games with simultaneous or alternating moves in a finite or infinite horizon under some suitable conditions. We adopt the general forms of intertemporal utilities, requiring neither stationarity nor additive separability.[3]

For deterministic games with perfect information (i.e., the players move alternately), the existence of pure-strategy subgame-perfect equilibria was shown in Börgers (1989, 1991), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), and Hellwig et al. (1990) with the model parameters being continuous in actions, extending the early work of Zermelo (1913).[4] However, if the "deterministic" assumption is dropped by introducing a passive player, Nature, then a pure-strategy subgame-perfect equilibrium need not exist as shown by a four-stage game in Harris et al. (1995). In fact, the nonexistence of a mixed-strategy subgame-perfect equilibrium in a five-stage alternating move game was provided by Luttmer and Mariotti (2003).[5] Thus, it is still an open problem to show the existence of (pure or mixed strategy) subgame-perfect equilibria in (finite or infinite horizon) perfect-information dynamic games with uncertainty under some general conditions.

Continuous dynamic games with almost perfect information in the sense that the players move simultaneously are considered in Harris et al. (1995). In such games, there are a finite number of active players and a passive player (Nature), and all the relevant model parameters are assumed to be continuous in both action and state variables (i.e.,

[1]Without the continuity at infinity condition, subgame-perfect equilibria may not exist in infinite-horizon dynamic games with finitely many actions; see Solan and Vieille (2003) for a counterexample.

[2]As noted in Section 2 of Harris et al. (1995), such a dynamic game with continuous choices provides a minimal nontrivial counterexample; see also Exercise 13.4 in Fudenberg and Tirole (1991) for another counterexample of Harris.

[3]To be specific, we assume that players's payoffs are functions of the whole histories endowed with the product topology, which does not need to be the discounted summation of the stage payoffs. For details, see Section 2.

[4]Alós-Ferrer and Ritzberger (2016) considered an alternative formulation of dynamic games with perfect information and without Nature, and showed the existence of subgame-perfect equilibrium.

[5]All those counterexamples show that various issues arise when one considers general dynamic games; see, for example, the discussions in Stinchcombe (2005). In the setting with incomplete information, even the equilibrium notion needs to be carefully treated; see Myerson and Reny (2019).

Nature's moves). It was shown in Harris et al. (1995) that subgame-perfect equilibria exist in those extended games obtained from the original games by introducing stagewise public randomization as a correlation device.[6] As mentioned above, they also demonstrated the possible nonexistence of subgame-perfect equilibrium through a simple example.

The first aim of this paper is to resolve the above two open problems (in both finite and infinite horizons) for the class of continuous dynamic games. We assume the state transition in each period (except for those periods with one active player) to be an atomless probability measure for any given history.[7] In Theorems 1 and 2 (and Proposition 1), we present the existence results for subgame-perfect equilibria, and also present some regularity properties of the equilibrium payoff correspondences, including compactness and upper hemicontinuity.[8] Note that our model allows the state history to fully influence all the model parameters, and hence, covers the case with stagewise public randomization in the sense that the state transition has an additional atomless component that is independently and identically distributed across time, and does not enter the payoffs, state transitions, and action correspondences. As a result, we obtain the existence result in Harris et al. (1995) as a special case. In addition, we also provide a new existence result for continuous stochastic games in Proposition 2; see Remark 3 for discussions.

For dynamic games with almost perfect information, our results allow the players to take mixed strategies. However, for the special class of continuous dynamic games with perfect information,[9] we obtain the existence of *pure-strategy* subgame-perfect equilibria in Theorem 2. When Nature is present, there has been no general result on the existence of equilibria (even in mixed strategies) for continuous dynamic games with perfect information. Our Theorem 2 provides a new existence result in pure strategy, which extends the results of Börgers (1989), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), Hellwig et al. (1990), and Zermelo (1913) to the case when Nature is present.

The condition of atomless transitions is minimal in the particular sense that the existence results for continuous dynamic games may fail to hold if (a) the passive player, Nature, is not present in the model as shown in Harris et al. (1995), or (b) with the presence of Nature, the state transition is not atomless even at one point of history as shown in Luttmer and Mariotti (2003).

The second aim of this paper is to consider an important extension in which the relevant model parameters are assumed to be continuous in actions, but measurable

[6]See also Mariotti (2000) and Reny and Robson (2002).

[7]A probability measure on a separable metric space is atomless if every single point has measure zero.

[8]Such an upper hemicontinuity property in terms of correspondences of equilibrium payoffs, outcomes, or correlated strategies is the key for proving the relevant existence results in Börgers (1989, 1991), Harris (1985), Harris et al. (1995), Hellwig and Leininger (1987), Hellwig et al. (1990), and Mariotti (2000).

[9]Dynamic games with perfect information also have wide applications. For example, see Amir (1996) and Phelps and Pollak (1968) for an intergenerational bequest game, and see Goldman (1980) and Peleg and Yaari (1973) for intrapersonal games in which consumers have changing preferences.

in states.[10] In particular, we show the existence of a subgame-perfect equilibrium in a general dynamic game with almost perfect information under some suitable conditions on the state transitions. Theorems 3 and 4 below go beyond our results on continuous dynamic games by dropping the continuity requirement on the state variables.[11] We work with the condition that the state transition in each period (except for those periods with one active player) has a component with a suitable density function with respect to some atomless reference measure.

In Appendix A, we provide a complete proof of Theorem 1 and point out those changes that are needed to prove Theorem 2 and Proposition 1. We follow the standard three-step procedure in obtaining subgame-perfect equilibria of dynamic games, namely, backward induction, forward induction, and approximation of the infinite horizon by the finite horizon. Because we drop the stagewise public randomization, new technical difficulties arise in the proofs. The main purpose of the step of backward induction is to show that if the payoff correspondence at a given stage satisfies certain regularity properties, then the equilibrium payoff correspondence at the previous stage is upper hemicontinuous. We notice that the condition of atomless transitions suffices for this purpose, and hence the exogenous stagewise public randomization is not needed for this step. For the step of forward induction, we need to obtain strategies that are jointly measurable in history. When there is a public randomization device, the joint measurability follows from the measurable version of Skorokhod's representation theorem and implicit function theorem as in Harris et al. (1995) and Reny and Robson (2002), respectively. Here we need to work with the deep "measurable" measurable choice theorem of Mertens (2003).

In Appendix B, we prove Theorem 3 first, and then describe the changes needed to prove Theorem 4 and Proposition B.1. The proofs for the results in measurable dynamic games are much more difficult than those in the case of continuous dynamic games. In the backward induction step, we obtain a new existence result for discontinuous games with stochastic endogenous sharing rules that extends the main result of Simon and Zame (1990) by allowing the payoff correspondence to be measurable (instead of upper hemicontinuous) in states.[12] So as to extend the results to the infinite-horizon setting,

[10]Since the agents need to make optimal choices, the continuity assumption in terms of actions is natural and widely adopted. However, the state variable is not a choice variable, and thus, it is unnecessary to impose the state continuity requirement in a general model. Note that the state measurability assumption is the minimal regularity condition one would expect for the model parameters. We may also point out that the proof for the case with state continuity in Appendix A is much simpler than the proof for the general case in Appendix B. For discussions on subgame-perfect ϵ-equilibria in dynamic games without the continuity conditions in actions, see Solan and Vieille (2003), Flesch et al. (2010), Laraki et al. (2013), Flesch and Predtetchinski (2016), and the references therein.

[11]In Appendix B, we also present a new existence result on subgame-perfect equilibria for a general stochastic game.

[12]In Simon and Zame (1990), the payoff is assumed to be a correspondence that is bounded and upper hemicontinuous, with nonempty, convex, and compact values. Note that the upper hemicontinuity condition on a correspondence is equivalent to the fact that the lower inverse of any closed set is closed; see Aliprantis and Border (2006, Lemma 17.4). Furthermore, the measurability condition on a correspondence means that the lower inverse of any closed set is measurable (see Appendix A.1). Thus, an upper hemicontinuous correspondence is automatically measurable. For more discussions of the approach in Simon and Zame (1990), see Harris et al. (2005) and Stinchcombe (2005).

we need to handle various subtle measurability issues due to the lack of continuity in the state variables in the more general model, which is the most difficult part of the proof for Theorem 3. [13]

The rest of the paper is organized as follows. The model is presented in Section 2. In Section 3, we provide a variation of the counterexample in Luttmer and Mariotti (2003) to demonstrate the key issues. The results for continuous dynamic games are given in Section 4. Section 5 extends continuous dynamic games to the setting in which the model parameters may only be measurable in the state variables. The proofs for the results of continuous dynamic games and measurable dynamic games are left to Appendices A and B, respectively. (Appendix B is available in a supplementary file on the journal website, http://econtheory.org/supp/2927/supplement.pdf.)

2. Model

In this section, we present the model for an infinite-horizon dynamic game with almost perfect information.

The set of players is $I_0 = \{0, 1, \dots, n\}$, where the players in $I = \{1, \dots, n\}$ are active and player 0 is Nature. All the players move simultaneously. Time is discrete and is indexed by $t = 0, 1, 2, \dots$.

The set of starting points is a product space $H_0 = X_0 \times S_0$, where X_0 is a compact metric space and S_0 is a Polish space (i.e., a complete separable metric space).[14] At stage $t \geq 1$, player i's action will be chosen from a subset of a Polish space X_{ti} for each $i \in I$, and $X_t = \prod_{i \in I} X_{ti}$. Nature's action is chosen from a Polish space S_t. Let $X^t = \prod_{0 \leq k \leq t} X_k$ and $S^t = \prod_{0 \leq k \leq t} S_k$. The Borel σ-algebras on X_t and S_t are denoted by $\mathcal{B}(X_t)$ and $\mathcal{B}(S_t)$, respectively. Given $t \geq 0$, a history up to stage t is a vector[15]

$$h_t = (x_0, s_0, x_1, s_1, \dots, x_t, s_t) \in X^t \times S^t.$$

The set of all such possible histories is denoted by H_t. For any $t \geq 0$, $H_t \subseteq X^t \times S^t$.

For any $t \geq 1$ and $i \in I$, let A_{ti} be a measurable, nonempty and compact correspondence[16] from H_{t-1} to X_{ti} such that $A_{ti}(h_{t-1})$ is the set of available actions for player $i \in I$ given the history h_{t-1}. Let $A_t = \prod_{i \in I} A_{ti}$. Then $H_t = \mathrm{Gr}(A_t) \times S_t$, where $\mathrm{Gr}(A_t)$ is the graph of A_t.

For any $x = (x_0, x_1, \dots) \in X^\infty$, let $x^t = (x_0, \dots, x_t) \in X^t$ be the truncation of x up to the period t. Truncations for $s \in S^\infty$ can be defined similarly. Let H_∞ be the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$. Then H_∞ is the set of

[13] Because our relevant model parameters are only measurable in the state variables, the usual method of approximating a limit continuous dynamic game by a sequence of finite games, as used in Börgers (1991), Harris et al. (1995) and Hellwig et al. (1990), is not applicable in this setting.

[14] In each stage $t \geq 1$, there will be a set of action profiles X_t and a set of states S_t. Without loss of generality, we assume that the set of initial points is also a product space for notational consistency.

[15] By abusing notation, we also view $h_t = (x_0, s_0, x_1, s_1, \dots, x_t, s_t)$ as the vector $(x_0, x_1, \dots, x_t, s_0, s_1, \dots, s_t)$ in $X^t \times S^t$.

[16] Suppose that Y and Z are both Polish spaces, and that Ψ is a correspondence from Y to Z. Hereafter, the measurability of Ψ, unless specifically indicated, is assumed to be the weak measurability with respect to the Borel σ-algebra $\mathcal{B}(Y)$ on Y. For definitions and detailed discussions, see Appendix A.1.

all possible histories in the game.[17] Hereafter, let H_∞ be endowed with the product topology. For any $t \geq 1$, Nature's action is given by a Borel measurable mapping f_{t0} from the history H_{t-1} to $\mathcal{M}(S_t)$, where $\mathcal{M}(S_t)$ denotes the set of all Borel probability measures on S_t and is endowed with the topology of weak convergence of measures on S_t.

For each $i \in I$, the payoff function u_i is a bounded Borel measurable mapping from H_∞ to $\mathbb{R}_{++}$. Without loss of generality, we can assume that the payoff function u_i is bounded from above by some $\gamma > 0$ for each $i \in I$.

For player $i \in I$, a strategy f_i is a sequence $\{f_{ti}\}_{t\geq 1}$ such that f_{ti} is a Borel measurable mapping from H_{t-1} to $\mathcal{M}(X_{ti})$ with $f_{ti}(A_{ti}(h_{t-1})|h_{t-1}) = 1$ for all $h_{t-1} \in H_{t-1}$. That is, player i can only take the mixed strategy concentrated on the available set of actions $A_{ti}(h_{t-1})$ given the history h_{t-1}. A strategy profile $f = \{f_i\}_{i\in I}$ is a combination of strategies of all active players.

In any subgame, a strategy combination generates a probability distribution over the set of possible histories. This probability distribution is called the path induced by the strategy combination in this subgame. Before describing (in Definition 1) how a strategy combination induces a path, we need to define some technical terms. Given a strategy profile $f = \{f_i\}_{i\in I}$, denote $\bigotimes_{i\in I_0} f_{(t'+1)i}$ as a transition probability from the set of histories $H_{t'}$ to $\mathcal{M}(X_{t'+1})$. For notational simplicity later on, we assume that $\bigotimes_{i\in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ represents the strategy profile in stage $t'+1$ for a given history $h_{t'} \in H_{t'}$, where $\bigotimes_{i\in I_0} f_{(t'+1)i}(\cdot|h_{t'})$ is the product of the probability measures $f_{(t'+1)i}(\cdot|h_{t'})$, $i \in I_0$. If λ is a finite measure on X and ν is a transition probability from X to Y, then $\lambda \diamond \nu$ is a measure on $X \times Y$ such that $\lambda \diamond \nu(A \times B) = \int_A \nu(B|x)\lambda(dx)$ for any measurable subsets $A \subseteq X$ and $B \subseteq Y$.

Definition 1. Suppose that a strategy profile $f = \{f_i\}_{i\in I}$ and a history $h_t \in H_t$ are given for some $t \geq 0$. Let $\tau_t = \delta_{h_t}$, where δ_{h_t} is the probability measure concentrated at the point h_t. If $\tau_{t'} \in \mathcal{M}(H_{t'})$ has already been defined for some $t' \geq t$, then let

$$\tau_{t'+1} = \tau_{t'} \diamond \left(\bigotimes_{i\in I_0} f_{(t'+1)i}\right).$$

Finally, let $\tau \in \mathcal{M}(H_\infty)$ be the unique probability measure on H_∞ such that $\mathrm{Marg}_{H_{t'}}\, \tau = \tau_{t'}$ for all $t' \geq t$. Then τ is called the path induced by f in the subgame h_t. For all $i \in I$, $\int_{H_\infty} u_i \, d\tau$ is the payoff of player i in this subgame.

3. An example

As mentioned in the Introduction, Luttmer and Mariotti (2003) presented a simple five-stage alternating move game that does not possess any subgame-perfect equilibrium. Below, we modify their counterexample to illustrate what could go wrong in a continuous dynamic game and we use this example to demonstrate some key issues.

Fix $0 \leq \epsilon \leq 1$. The game G_ϵ proceeds in five stages.

[17] A finite-horizon dynamic game can be regarded as a special case of an infinite-horizon dynamic game in the sense that the action correspondence A_{ti} is point-valued for each player $i \in I$ and $t \geq T$ for some stage $T \geq 1$; see, for example, Börgers (1989) and Harris et al. (1995).

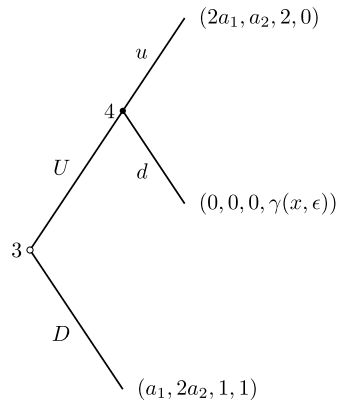

FIGURE 1. The subgame (a_1, a_2, x).

- In stage 1, player 1 chooses an action $a_1 \in [0, 1]$.
- In stage 2, player 2 chooses an action $a_2 \in [0, 1]$.
- In stage 3, Nature chooses some $x \in [-2 - \epsilon + a_1 + a_2, 2 + \epsilon - a_1 - a_2]$ based on the uniform distribution $\eta^{\epsilon}_{(a_1,a_2)}$.
- After Nature's choice, players 3 and 4 move sequentially. The subgame that follows a history (a_1, a_2, x) and the associated payoffs for all four active players is shown in Figure 1, where

$$\gamma(x, \epsilon) = \begin{cases} x + \epsilon, & \text{if } x < -\epsilon, \\ x - \epsilon, & \text{if } x > \epsilon, \\ 0, & x \in [-\epsilon, \epsilon]. \end{cases}$$

In the following discussion, let α and β be the probabilities with which players 3 and 4 choose U and u, respectively. Consider a subgame (a_1, a_2, x). Let $\tilde{P}^{\epsilon}_3(a_1, a_2, x)$ (resp. $\tilde{P}^{\epsilon}_2(a_1, a_2)$) be the set of expected payoffs for players 1 and 2 in stage 3 (resp. stage 2).[18]

- If $x < -\epsilon$, then the equilibrium continuation path is (U, u) (i.e., $\alpha = 1$ and $\beta = 1$) and $\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \{(2a_1, a_2)\}$.
- If $x > \epsilon$, then the equilibrium continuation path is D (i.e., $\alpha = 0$ and $\beta = 0$) and $\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \{(a_1, 2a_2)\}$.
- If $x \in [-\epsilon, \epsilon]$, then the set of equilibrium continuation paths is characterized by three segments of mixing probabilities: $\alpha = 0$ and $\beta \in [0, \frac{1}{2}]$; $\alpha \in [0, 1]$ and $\beta = \frac{1}{2}$; $\alpha = 1$ and $\beta \in [\frac{1}{2}, 1]$. Then

$$\tilde{P}^{\epsilon}_3(a_1, a_2, x) = \left\{\left(a_1, \left(2 - \frac{3}{2}\alpha\right)a_2\right) | \alpha \in [0, 1]\right\} \cup \left\{(2a_1\beta, a_2\beta) | \beta \in \left[\frac{1}{2}, 1\right]\right\},$$

[18]For simplicity, we focus on the equilibrium payoffs of players 1 and 2.

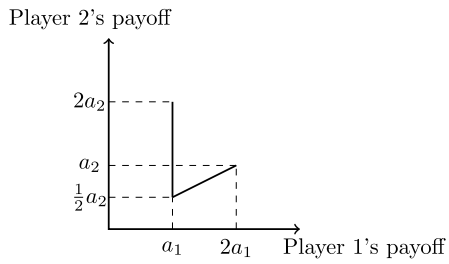

FIGURE 2. D.

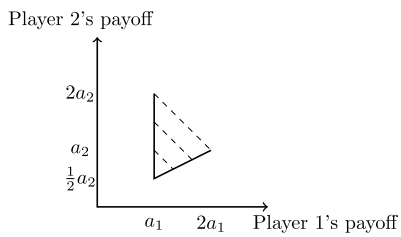

FIGURE 3. $\mathrm{Co}(D)$.

which is not convex when $a_1 > 0$ and $a_2 > 0$.

Denote $D = \{(a_1, (2 - \frac{3}{2}\alpha)a_2) | \alpha \in [0, 1]\} \cup \{(2a_1\beta, a_2\beta) | \beta \in [\frac{1}{2}, 1]\}$ and $\mathrm{Co}(D)$ as the convex hull of the set D. Below, for $a_1 > 0$ and $a_2 > 0$, D is the union of the two segments in Figure 2 and $\mathrm{Co}(D)$ is the dashed area with boundaries in Figure 3.

Fix any $\epsilon > 0$. Nature's move x is uniformly distributed on the nondegenerate interval $[-2 - \epsilon + a_1 + a_2, 2 + \epsilon - a_1 - a_2]$, which is symmetric around zero for any (a_1, a_2). The correspondence $\tilde{P}_3^\epsilon(a_1, a_2, x)$ is upper hemicontinuous, but is not convex-valued when $x \in [-\epsilon, \epsilon]$ and $a_1, a_2 > 0$ (the set D is not convex in this case). Given any (a_1, a_2), the set of expected equilibrium continuation payoffs for players 1 and 2 is[19]

$$\begin{aligned}
\tilde{P}_2^\epsilon(a_1, a_2) &= \int_{-3}^{3} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta_{(a_1,a_2)}^\epsilon(dx) \\
&= \int_{-3}^{-\epsilon} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta_{(a_1,a_2)}^\epsilon(dx) + \int_{-\epsilon}^{\epsilon} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta_{(a_1,a_2)}^\epsilon(dx) \\
&\quad + \int_{\epsilon}^{3} \tilde{P}_3^\epsilon(a_1, a_2, x)\eta_{(a_1,a_2)}^\epsilon(dx) \\
&= \frac{\epsilon}{2 + \epsilon - a_1 - a_2}\mathrm{Co}(D) + \frac{2 - a_1 - a_2}{2 + \epsilon - a_1 - a_2}\left\{\left(\frac{3}{2}a_1, \frac{3}{2}a_2\right)\right\} \\
&= \int_{-3}^{-\epsilon} \mathrm{Co}\big(\tilde{P}_3^\epsilon(a_1, a_2, x)\big)\eta_{(a_1,a_2)}^\epsilon(dx) + \int_{-\epsilon}^{\epsilon} \mathrm{Co}\big(\tilde{P}_3^\epsilon(a_1, a_2, x)\big)\eta_{(a_1,a_2)}^\epsilon(dx)
\end{aligned}$$

[19]Given two sets $D_1, D_2 \subseteq \mathbb{R}^l$, $D_1 + D_2 = \{d_1 + d_2 : d_i \in D_i, i = 1, 2\}$; for $c \in \mathbb{R}$, $cD_1 = \{cd_1 : d_1 \in D_1\}$.

$$+\int_{\epsilon}^{3} \mathrm{Co}\big(\tilde{P}_3^{\epsilon}(a_1, a_2, x)\big)\eta_{(a_1,a_2)}^{\epsilon}(dx)$$
$$= \int_{-3}^{3} \mathrm{Co}\big(\tilde{P}_3^{\epsilon}(a_1, a_2, x)\big)\eta_{(a_1,a_2)}^{\epsilon}(dx).$$

For $x \in [-2-\epsilon+a_1+a_2, -\epsilon)$ or $(\epsilon, 2+\epsilon-a_1-a_2]$, $\tilde{P}_3^{\epsilon}(a_1, a_2, x)$ is a singleton and, hence, is convex-valued and coincides with $\mathrm{Co}(\tilde{P}_3^{\epsilon}(a_1, a_2, x))$. For $x \in [-\epsilon, \epsilon]$, $\tilde{P}_3^{\epsilon}(a_1, a_2, x)$ is the set D. Its integration on $[-\epsilon, \epsilon]$ under the uniform distribution is simply $\mathrm{Co}(D)$, and hence it coincides with $\mathrm{Co}(\tilde{P}_3^{\epsilon}(a_1, a_2, x))$. It is also clear that $\tilde{P}_2^{\epsilon}(a_1, a_2)$ is upper hemicontinuous in $(a_1, a_2) \in [0, 1] \times [0, 1]$.

There are two general observations: (a) the integral of a correspondence coincides with the integral of the convex hull of the correspondence based on an atomless measure, and (b) the integral of a convex-valued, upper hemicontinuous correspondence based on a continuous transition probability is still upper hemicontinuous.[20] As a result, the integral of an upper hemicontinuous correspondence is still upper hemicontinuous and convex-valued based on an *atomless* continuous transition probability.[21] In the particular case of this example, the above paragraph shows that even though $\tilde{P}_3^{\epsilon}(a_1, a_2, x)$ is not always convex-valued on the nondegenerate set $[-\epsilon, \epsilon]$, $\tilde{P}_2^{\epsilon}(a_1, a_2)$ is still convex-valued and upper hemicontinuous. Such a result also follows from the general observations since Nature's move $\eta_{(a_1,a_2)}^{\epsilon}$ is atomless and continuous in (a_1, a_2). Here is a pure-strategy subgame-perfect equilibrium in the game G_ϵ for $\epsilon > 0$: players 1 and 2 choose $a_1 = 1$ and $a_2 = 1$, players 3 and 4 choose U and u when $x < 0$, and choose D and d when $x \geq 0$. In this equilibrium, both players 1 and 2 get the payoff $\frac{3}{2}$.

For the case $\epsilon = 0$, the game G_0 is the counterexample in Luttmer and Mariotti (2003), which does not have any subgame-perfect equilibrium. If $a_1 + a_2 < 2$, then Nature's move x is uniformly distributed on the nondegenerate interval $[-2 + a_1 + a_2, 2 - a_1 - a_2]$. As $x = 0$ is drawn with probability 0, the nonconvexity of the set of continuation payoffs for players 1 and 2 at $x = 0$ does not matter. The expected continuation payoffs for players 1 and 2 are $\frac{3}{2}a_1$ and $\frac{3}{2}a_2$, respectively. That is, $\tilde{P}_2^0(a_1, a_2) = \{(\frac{3}{2}a_1, \frac{3}{2}a_2)\}$ when $a_1 + a_2 < 2$. If $a_1 + a_2 = 2$ (i.e., $a_1 = a_2 = 1$), then Nature's move must be $x = 0$ and, hence,

$$\tilde{P}_2^0(1, 1) = \tilde{P}_3^0(1, 1, 0) = \left\{\left(1, \left(2 - \frac{3}{2}\alpha\right)\right)\middle|\alpha \in [0, 1]\right\} \cup \left\{(2\beta, \beta)|\beta \in \left[\frac{1}{2}, 1\right]\right\}.$$

Whenever $a_1 + a_2 < 2$, both players 1 and 2 have the incentive to choose their actions as close to 1 as possible, which gives them the expected payoff arbitrarily close to $\frac{3}{2}$. However, when both players 1 and 2 choose the action 1, some of them get a payoff no more than 1. This implies that there does not exist any subgame-perfect equilibrium.

As shown above, for any $\epsilon > 0$, both players 1 and 2 in the game G_ϵ have $\frac{3}{2}$ as their equilibrium payoffs. Since $(\frac{3}{2}, \frac{3}{2})$ cannot be the equilibrium payoffs of players 1 and 2 in the game G_0, the equilibrium payoff correspondence of the games G_ϵ, $\epsilon \geq 0$, is not upper hemicontinuous at $\epsilon = 0$.

[20] See Lemma 7 in Appendix A.1.

[21] Note that when the transition probability has an atom in its values, both properties may not be true. This is demonstrated in the case $\epsilon = 0$ below.

Note that Nature's move $\eta^0_{(a_1,a_2)}$ is continuous in $(a_1, a_2) \in [0,1] \times [0,1]$ and is atomless except for the one point $(a_1, a_2) = (1,1)$. Footnote 21 indicates that the integral of an upper hemicontinuous correspondence with respect to such a transition probability may not be upper hemicontinuous and convex-valued. Indeed, even though $\tilde{P}^0_3$ is an upper hemicontinuous correspondence, $\tilde{P}^0_2$ is neither upper hemicontinuous nor convex at the point $(1,1)$. In particular, $(\frac{3}{2}a_1, \frac{3}{2}a_2) \in \tilde{P}^0_2(a_1, a_2)$ when $a_1 + a_2 < 2$, while its limit $(\frac{3}{2}, \frac{3}{2}) \notin \tilde{P}^0_2(1,1)$ when both a_1 and a_2 converge to 1.

4. Continuous dynamic games

In this section, we consider continuous dynamic games in the sense that all the model parameters (the payoff functions, state transitions, and action correspondences) are continuous in both action and state variables. We show that subgame-perfect equilibria exist for continuous dynamic games under the condition of atomless transitions. In Section 4.1, we first consider dynamic games with almost perfect information and show the existence of subgame-perfect equilibria. In Section 4.2, we consider dynamic games with perfect information in the sense that players move sequentially, and prove the existence of *pure-strategy* subgame-perfect equilibria. In Section 4.3, we provide a road map for proving Theorems 1 and 2; the details of the proofs are left to Appendix A. In Section 4.4, we extend the model so that the previous existence results for continuous dynamic games with perfect and almost perfect information are covered as special cases. As a by-product, we provide a new existence result for continuous stochastic games.

4.1 *Continuous dynamic games with almost perfect information*

In this subsection, we study an infinite-horizon continuous dynamic game with almost perfect information. Intuitively, we work with the class of games in which all the relevant parameters of the game, including action correspondences, Nature's move, and payoff functions, vary smoothly with respect to the state and action variables. In particular, a dynamic game is said to be *continuous* if for each t and i, the following statements hold.

(i) The action correspondence A_{ti} is continuous on H_{t-1}.[22]

(ii) The transition probability f_{t0} is a continuous mapping from H_{t-1} to $\mathcal{M}(S_t)$, where $\mathcal{M}(S_t)$ is endowed with the topology of weak convergence (also called the weak star topology); that is, for any bounded continuous function ψ on S_t, the integral

$$\int_{S_t} \psi(s_t) f_{t0}(ds_t | h_{t-1})$$

is continuous in h_{t-1}.

(iii) The payoff function u_i is continuous on H_∞.

[22]A correspondence is said to be continuous if it is both upper hemicontinuous and lower hemicontinuous. For definitions and detailed discussion, see Appendix A.1.

Below we propose the condition of "atomless transitions" on the state space, which means that Nature's move is an atomless probability measure in any stage.

Assumption 1 (Atomless Transitions). *For each $t \geq 1$, $f_{t0}(h_{t-1})$ is an atomless Borel probability measure for each $h_{t-1} \in H_{t-1}$.*

The notion of subgame-perfect equilibrium is given below. It requires each player's strategy to be optimal in every subgame given the strategies of all other players.

Definition 2 (Subgame-Perfect Equilibrium). A subgame-perfect equilibrium (SPE) is a strategy profile f such that for all $i \in I$, $t \geq 0$, and all $h_t \in H_t$, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy.

Let $E_t(h_{t-1})$ be the set of subgame-perfect equilibrium payoffs in the subgame h_{t-1}. The following result shows that a subgame-perfect equilibrium exists, and the equilibrium correspondence E_t satisfies certain desirable compactness and upper hemicontinuity properties.

Theorem 1. *If a continuous dynamic game with almost perfect information has atomless transitions, then it possesses a subgame-perfect equilibrium. In addition, E_t is nonempty and compact valued, and upper hemicontinuous on H_{t-1} for any $t \geq 1$.*

Remark 1. Theorem 1 goes beyond the main result of Harris et al. (1995) for continuous dynamic games, where the existence of subgame-perfect equilibria was shown for those extended games obtained from the original games by introducing stagewise public randomization as a correlation device. Such a correlation device does not influence the payoffs, transitions, or action correspondences. It is clear that the extended games with stagewise public randomization as in Harris et al. (1995) automatically satisfy the condition of atomless transitions. The states in our model are completely endogenous in the sense that they can affect all the model parameters such as payoffs, transitions, and action correspondences.

4.2 *Continuous dynamic games with perfect information*

In this subsection, we consider another important class of continuous dynamic games, namely continuous dynamic games with perfect information (with or without Nature). In such games, players move sequentially. We show the existence of pure-strategy subgame-perfect equilibria. In particular, the condition of atomless transitions is imposed only when Nature moves.

In a continuous dynamic game with perfect information, there is only one player moving in each stage. In stage t, if A_{ti} is not point-valued for some player $i \in I$, then A_{tj} is point-valued for any $j \in I$ as long as $j \neq i$, and $f_{t0}(h_{t-1}) \equiv \delta_{s_t}$ for some s_t; that is, only player i is active in stage t, while all the other players are inactive. If the state transition f_{t0} does not put probability 1 on some point, then A_{ti} must be point-valued for any $i \in I$; that is, only Nature can move in stage t and all the players $i \in I$ are inactive in this stage. A continuous dynamic game with perfect information is said to have atomless transitions if $f_{t0}(h_{t-1})$ is an atomless Borel probability measure when Nature moves only in stage t.

Theorem 2. *If a continuous dynamic game with perfect information has atomless transitions, then it possesses a pure-strategy subgame-perfect equilibrium. In addition, E_t is nonempty and compact valued, and upper hemicontinuous on H_{t-1} for any $t \geq 1$.*

Remark 2. As shown in Börgers (1989), Fudenberg and Levine (1983), Harris (1985), Hellwig and Leininger (1987), Hellwig et al. (1990), and Zermelo (1913), pure-strategy subgame-perfect equilibria exist in deterministic (i.e., without Nature) continuous dynamic games with perfect information. Theorem 2 extends those existence results to the case with Nature. We may point out that the condition of atomless transitions in Theorem 2 is minimal. In particular, the games in Section 3 can be viewed as an alternating move game with a starting point $\epsilon \in [0, 1]$, where the transition probability $\eta^{\epsilon}_{(a_1,a_2)}$ in the third period is continuous in $(\epsilon, a_1, a_2) \in [0, 1]^3$ and is atomless except for the one point $(\epsilon, a_1, a_2) = (0, 1, 1)$. The violation of our condition of atomless transitions at just one point leads to the failure of the conclusions of Theorem 2. [23]

4.3 *A road map for proving Theorems 1 and 2*

The existence results are established in three steps. The backward induction step aims to show that if the equilibrium payoff correspondence Q_t in stage t is well behaved (bounded, nonempty and compact valued, and upper hemicontinuous), then these desirable properties can be preserved for the equilibrium payoff correspondence Q_{t-1} in the previous stage $t-1$. As will be explained, the atomless transition condition plays an important role in this step. Next, given the equilibrium payoff correspondences across different periods, one needs to construct the equilibrium strategy profile stage by stage that is consistent with the equilibrium payoff correspondences. This is done in the forward induction step. The first two steps together prove the equilibrium existence results for finite-horizon dynamic games. The last step relates finite-horizon dynamic games to infinite-horizon dynamic games based on the condition of continuity at infinity. We sketch the main ideas of the proof based on simultaneous-move games and point out the modifications for alternating-move games whenever necessary.

Step 1. We explain the first (backward induction) step via a T-stage dynamic game. Let $Q_{(T+1)}(h_T)$ be the singleton set with one element vector $(u_1(h_T), \dots, u_n(h_T))$ for any T-stage history h_T, where u_i is the payoff function of player i at the last stage T (a bounded continuous function from the space of complete histories H_T to $\mathbb{R}$). Hence, $Q_{(T+1)i}$ is a bounded, nonempty and compact valued, and upper hemicontinuous correspondence. Given a history h_{T-1} at stage $T-1$, the state s_T at stage T follows the distribution $f_{T0}(\cdot|h_{T-1})$. For an action profile x_T and a state s_T at stage T, (h_{T-1}, x_T, s_T) is a history at stage T. Let

$$P_T(h_{T-1}, x_T) = \int_{S_T} Q_{T+1}(h_{T-1}, x_T, s_T) f_{T0}(ds_T|h_{T-1}).$$

Then $P_T(h_{T-1}, \cdot)$ is the set of expected possible payoff vectors in the subgame h_{T-1}.

[23] Remark 4 indicates that Theorem 2 can be generalized to the case when the state transitions either are atomless, or have the support inside a fixed finite set irrespective of the history at a particular stage.

Let $\Phi(Q_{T+1})(h_{T-1})$ be the set of all mixed-strategy Nash equilibrium payoffs for the game with the action set $A_{Ti}(h_{T-1})$ and the payoff function $P_T(h_{T-1}, \cdot)$. Then $\Phi(Q_{T+1})$ is a bounded, nonempty and compact valued, and upper hemicontinuous correspondence from H_{T-1} to $\mathbb{R}^n$. Intuitively, $\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1})$ represents the set of all possible payoff vectors in the subgame h_{T-2} when Nature's move is s_{T-1} and the players choose the action profile x_{T-1} in stage $T-1$.[24] Note that in the subgame h_{T-2},

$$P_{T-1}(h_{T-2}, x_{T-1}) = \int_{S_{T-1}} \Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(ds_{T-1}|h_{T-2})$$

is the set of payoff vectors. It is shown in Simon and Zame (1990) that if a payoff correspondence P_{T-1} is bounded, nonempty, convex and compact valued, and upper hemicontinuous, then it possesses a Borel (possibly discontinuous) selection such that there exists an equilibrium in mixed strategy by taking this selection as the payoff function.[25] The difficulties here are that (a) the correspondence P_{T-1} is no longer single-valued and may not be convex-valued; (b) even though $\Phi(Q_{T+1})$ is an upper hemicontinuous correspondence, it is not clear whether P_{T-1} is upper hemicontinuous.[26] By introducing the condition of "atomless transition," we show that (a) even though $\Phi(Q_{T+1})$ may not be convex-valued, if $f_{(T-1)0}(h_{T-2})$ is atomless, then

$$\begin{aligned}&\int_{S_{T-1}} \Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(ds_{T-1}|h_{T-2})\\ &\quad = \int_{S_{T-1}} \operatorname{co}\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, s_{T-1}) f_{(T-1)0}(ds_{T-1}|h_{T-2});\end{aligned}$$

hence, $P_{T-1}(h_{T-2}, x_{T-1})$ is convex; (b) $P_{T-1}(h_{T-2}, x_{T-1})$ is upper hemicontinuous because the correspondence $\operatorname{co}\Phi(Q_{T+1})$ under the integral is convex-valued and upper hemicontinuous. We can then repeat this backward induction argument from $\Phi(Q_{T+1})$ until the first stage.

Note that the key in the backward induction step is to preserve the convexity and upper hemicontinuity of the correspondences. As one arrives at the first stage, there is no need to conduct the backward induction again. Thus, our result can be strengthened by relaxing the condition of atomless transitions in the first stage. In simultaneous-move games, the argument in the previous paragraph requires that Nature be active and

[24]For $h_{T-1} = (h_{T-2}, x_{T-1}, s_{T-1})$, $\Phi(Q_{T+1})$ is a correspondence from H_{T-1} to $\mathbb{R}^n$. Given (h_{T-2}, x_{T-1}), we slightly abuse the notation by viewing $\Phi(Q_{T+1})(h_{T-2}, x_{T-1}, \cdot)$ as a correspondence from S_{T-1} to $\mathbb{R}^n$.

[25]It was demonstrated in Stinchcombe (2005, Example 2.2) that given an exogenous payoff correspondence with two measurable selections v and u, an equilibrium strategy of a player for the game with v as the payoff functions may be a strictly dominated strategy for the game with u as the payoff functions. Such an issue does not arise in our setting. Our primitives for the payoffs are the payoff functions (not payoff correspondences) of the players. When a full history is given, the players have a unique payoff vector in our setting. The payoff correspondence in our backward induction step is endogenous. As explained in Steps 2 and 3, each payoff vector given by the payoff correspondence corresponds to a subgame-perfect equilibrium in the original dynamic game. It is clear that any subgame-perfect equilibrium strategy of a player in a dynamic game cannot be a strictly dominated strategy of that game.

[26]As illustrated in Section 3, the upper hemicontinuity property may not be preserved if the transition probability has an atom in its value.

have an atomless transition in every stage (except the first stage). In alternating-move games, we require only that Nature's move be atomless whenever Nature is active (except the first stage). In particular, when Nature is inactive in some stage t, P_{t-1} is indeed $\Phi(Q_{t+1})$, a bounded, nonempty and compact valued, and upper hemicontinuous correspondence. The one who is the only active player in that stage faces a single-player decision problem. The key observation is that the only active player must possess an optimal choice in pure strategy even though P_{t-1} may not be convex-valued. When Nature is active, the reason why the backward induction argument holds is the same as that in the previous paragraph.

Step 2. We now describe the second (forward induction) step. Suppose that q_t is a measurable selection of $\Phi(Q_{t+1})$ in some stage t. Given the construction in the backward induction step, $q_t(h_{t-1})$ represents a possible payoff vector in the subgame h_{t-1} if all the players follow some equilibrium strategy in the subsequent stages. The aim of this step is to identify those subsequent equilibrium strategies and the corresponding payoff functions.

Based on the construction of $\Phi(Q_{t+1})$, one can expect that in every subgame h_{t-1}, there exists a strategy profile $f_t(h_{t-1})$ and a payoff profile $g_t(h_{t-1}, \cdot) \in P_t(h_{t-1}, \cdot)$ such that for all $h_{t-1} \in H_{t-1}$, (i) $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(dx|h_{t-1})$; (ii) $f_t(h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with the payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$.

The key technical difficulties here are that (a) the payoff function g_t needs to be jointly measurable in (h_{t-1}, x) and (b) one needs to further construct a jointly measurable selection q_{t+1} of Q_{t+1} (in (h_{t-1}, x_t, s_t)) such that $g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) \times f_{t0}(ds_t|h_{t-1})$ for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$. We solve the first issue by carefully modifying the argument in Reny and Robson (2002). For the second difficulty, we show that a deep "measurable" measurable choice theorem of Mertens (2003) can be used to address this issue.[27]

By completing this step, we establish the relationship between the equilibrium payoff correspondence in stage $t+1$ and the equilibrium payoff correspondence in stage t if all players play some equilibrium strategy in the subsequent stage. Together with the first step, the forward induction helps us obtain the equilibrium existence result in dynamic games with finite stages as follows. We can start with backward induction from the last period and stop at the initial period, and then run forward induction from the initial period to the last period.

Step 3. This step proves the equilibrium existence result in infinite-horizon dynamic games via Lemmas 11–15.

Since there is no last stage in the infinite-horizon setting, it is not clear where one should start with the backward induction argument. We pick an arbitrary stage $\tau \geq 1$ and let $Q^\tau_{\tau+1}$ be the expected payoff correspondence in stage τ if the players are free to choose any (not necessarily equilibrium) strategies in the future stages. Then run backward induction based on $Q^\tau_{\tau+1}$ from stage τ, and denote Q^τ_t as the equilibrium payoff correspondence in stage t for $t \leq \tau$. For $t \geq \tau + 1$, let $Q^\tau_t = Q^\tau_{\tau+1}$. Lemmas 11 and

[27]In Harris et al. (1995) and Reny and Robson (2002), the joint measurability follows from the measurable version of Skorokhod's representation theorem and implicit function theorem, respectively. These arguments are not applicable here.

12 show that the set of possible equilibrium payoff vectors satisfy desirable properties. In particular, Q_t^τ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.

It is easy to see that $Q_t^\tau(h_{t-1}) \subseteq Q_t^{\tau-1}(h_{t-1})$ for any h_{t-1}; that is, $\{Q_t^\tau\}_{\tau\geq 1}$ is a decreasing sequence in terms of τ. Denote $Q_t^\infty = \bigcap_{\tau\geq 1} Q_t^\tau$. Lemma 13 shows that $Q_t^\infty = \Phi(Q_{t+1}^\infty)$. By induction, $Q_t^\infty = \Phi^{\tau-t}(Q_\tau^\infty)$ for any $\tau > t$; that is, given the payoff correspondence Q_τ^∞ in stage τ for $\tau > t$, Q_t^∞ is the equilibrium payoff correspondence in stage t due to the construction of backward induction. Because of the assumption of continuity at infinity, the strategies in the far future are not important. For fixed t, it means that Q_t^∞ will be very close to the set by running backward induction from stage τ to stage t based on the actual equilibrium payoff correspondence in stage τ if τ is sufficiently large. Since Q_t^∞ is the intersection of all such Q_τ^∞, it is natural to expect that Q_t^∞ is indeed the equilibrium payoff correspondence in stage t. Recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame h_{t-1}. Given a measurable selection c_t of $\Phi(Q_{t+1}^\infty)$, Lemma 14 shows that $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector in the subgame h_{t-1} by constructing the subsequent equilibrium strategies based on the forward induction; that is, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq E_t(h_{t-1})$. In Lemma 15, we show that $E_t(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$. Then we have $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$. This completes the sketch for the infinite-horizon case.

4.4 *An extension*

In this subsection, we extend the model of continuous dynamic games as specified in the previous two subsections. The aim is to combine the models of dynamic games with perfect and almost perfect information, and cover an important class of dynamic games, namely stochastic games. We show the existence of a subgame-perfect equilibrium such that whenever there is only one active player at some stage, the player can play pure strategy as part of the equilibrium strategies. As a by-product, we obtain a new existence result for continuous stochastic games.

For concreteness, we allow for the case in which (a) the state transition depends on the action profile in the current stage as well as on the previous history, and (b) the players may have perfect information in some stages. The first modification covers the model of stochastic games as a special case. The second change allows us to combine the models of dynamic games with perfect and almost perfect information.

Step 1. For each $t \geq 1$, the choice of Nature depends not only on the history h_{t-1}, but also on the action profile x_t in this stage. For any $t \geq 1$, suppose that A_{t0} is a continuous, nonempty and closed valued correspondence from $\mathrm{Gr}(A_t)$ to S_t. Then $H_t = \mathrm{Gr}(A_{t0})$, and H_∞ is the subset of $X^\infty \times S^\infty$ such that $(x, s) \in H_\infty$ if $(x^t, s^t) \in H_t$ for any $t \geq 0$.

Step 2. Nature's action is given by a continuous mapping f_{t0} from $\mathrm{Gr}(A_t)$ to $\mathcal{M}(S_t)$ such that $f_{t0}(A_{t0}(h_{t-1}, x_t)|h_{t-1}, x_t) = 1$ for all $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.

Step 3. For each $t \geq 1$, we use the notation N_t to track whether there is a unique active player in stage t. In particular, let

$$N_t = \begin{cases} 1, & \text{if } f_{t0}(h_{t-1}, x_t) \equiv \delta_{s_t} \text{ for some } s_t \text{ and} \\ & \big|\{i \in I\colon A_{ti} \text{ is not point-valued}\}\big| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $|K|$ represents the number of points in the set K. Thus, if $N_t = 1$, then the player who is active in the period t is the only active player and has perfect information. If $N_t = 0$, then Nature moves in this stage.

Similarly as in Section 4.2, we can drop the condition of atomless transition in those periods with only one active player in I.

Assumption 2 (Atomless Transitions′). *(i) For any $t \geq 1$ with $N_t = 1$, S_t is a singleton set $\{\acute{s}_t\}$.*

(ii) For each $t \geq 1$ with $N_t = 0$, $f_{t0}(h_{t-1}, x_t)$ is an atomless Borel probability measure for each $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$.

The result on the equilibrium existence is presented below.

Proposition 1. *If a continuous dynamic game (as described above) satisfies the condition of atomless transitions′, then it possesses a subgame-perfect equilibrium f. In particular, for $j \in I$ and $t \geq 1$ such that $N_t = 1$ and player j is the only active player in this period, f_{tj} can be chosen to be deterministic. In addition, E_t is nonempty and compact valued, and upper hemicontinuous on H_{t-1} for $t \geq 1$.*

As the extension above covers the model of continuous stochastic games, a new equilibrium existence result can be stated below for continuous stochastic games.

Proposition 2. *If a continuous stochastic game has atomless transitions′, then it possesses a subgame-perfect equilibrium.*

Remark 3. Consider a standard stochastic game with uncountable states as in Mertens and Parthasarathy (2003), where the existence of a subgame-perfect equilibrium was shown by assuming the state transitions (not necessarily atomless) to be norm continuous (in the norm topology on the space of Borel measures) with respect to the actions in the previous stage.[28] It is noted in Maitra and Sudderth (2007 p. 712) that "[t]his is a very strong condition." Maitra and Sudderth (2007) also indicated on the same page the desirability to weaken such a norm continuity condition: "it would be preferable to assume some sort of weak continuity...." By restricting our result on general dynamic games to the setting of stochastic games, we obtain the existence of subgame-perfect equilibria in stochastic games whose state transitions are atomless and continuous in

[28]For detailed discussions on general stochastic games, see Jaśkiewicz and Nowak (2016).

the weak star topology on the space of Borel measures. Our result cannot be covered by the result in Mertens and Parthasarathy (2003) and vice versa.

We provide a simple example to demonstrate that our continuity condition on the state transitions is indeed weaker than the norm continuity condition required by Mertens and Parthasarathy (2003). Consider the following state transitions.

- The action space is $A_1 = A_2 = [0, 1]$.
- The state space in stage t is a product space: $S_t = S_{t1} \times S_{t2} = [0, 1] \times [0, 1]$.
- Given $s_{t-1} = (s_{(t-1)1}, s_{(t-1)2})$ and $a_{t-1} = (a_{(t-1)1}, a_{(t-1)2})$, the stage transition $f_{t0}(\cdot|s_{t-1}, a_{t-1})$ induces a product probability measure $\psi_{t1}(\cdot|s_{t-1}, a_{t-1}) \otimes \psi_{t2}(\cdot|s_{t-1}, a_{t-1})$ on $S_{t1} \times S_{t2}$, where $\psi_{t2}(\cdot|s_{t-1}, a_{t-1})$ is the uniform distribution on $S_{t2} = [0, 1]$ regardless of (s_{t-1}, a_{t-1}), and

$$\psi_{t1}\left(s_{t1} = \frac{s_{(t-1)1} + s_{(t-1)2} + a_{(t-1)1} + a_{(t-1)2}}{4} | s_{t-1}, a_{t-1}\right) = 1;$$

  that is, given (s_{t-1}, a_{t-1}), $\psi_{t1}(\cdot|s_{t-1}, a_{t-1})$ puts probability 1 on their average.

Since ψ_{t2} gives the uniform distribution on S_{t2}, the state transition is atomless. In addition, the state transition f_{t0} is continuous as ψ_{t1} is continuous and ψ_{t2} is constant.

The state transition f_{t0} is clearly not norm continuous. For example, we fix $s_{t-1} = (s_{(t-1)1}, s_{(t-1)2})$ and a sequence $\{a^n_{t-1}\}_{n\geq 0}$ such that $s_{(t-1)1} + s_{(t-1)2} = \frac{1}{2}$, $(a^n_{(t-1)1}, a^n_{(t-1)2}) \to (a^0_{(t-1)1}, a^0_{(t-1)2})$ as $n \to \infty$, and $a^n_{(t-1)1} + a^n_{(t-1)2} = \frac{1}{2} - \frac{1}{2n}$ for any $n \geq 1$. Then

$$f_{t0}\left(\left\{\frac{1}{4}\right\} \times [0, 1]|s_{t-1}, a^n_{t-1}\right) = 0 \nrightarrow 1 = f_{t0}\left(\left\{\frac{1}{4}\right\} \times [0, 1]|s_{t-1}, a^0_{t-1}\right).$$

5. Measurable dynamic games

In this section, we consider the more general setting in which the model parameters are jointly measurable in the action and state variables, but continuity is only required for the action variables. The proofs of the results in this section are left to Appendix B.

In Section 5.1, we adopt the model specified in Section 4.1, but relax the continuity requirement to measurability in the state variables. To obtain the existence of subgame-perfect equilibria, we strengthen the condition "atomless transitions" to the condition "atomless reference measure" (ARM) on the state transitions. The latter condition means that in each stage, there is an atomless reference measure and the state transitions are absolutely continuous with respect to this reference measure. In Section 5.2, we consider dynamic games with perfect information. The ARM-type condition is imposed only when Nature moves. We show the existence of *pure-strategy* subgame-perfect equilibria. In Section 5.3, we provide a road map for proving Theorems 3 and 4. To omit the repetitive descriptions, we follow the argument in Section 4.3 and highlight only the necessary changes.

In Appendix B, we present a further extension by partially relaxing the ARM condition in two ways. First, we allow the possibility that there is only one active player

(but no Nature) at some stages, where the ARM-type condition is dropped. Second, we introduce an additional weakly continuous component on the state transitions at any other stages. In addition, we allow the state transition in each period to depend on the current actions as well as on the previous history. As the generalization of the model in Section 4.4, (a) we combine the models for measurable dynamic games with perfect and almost perfect information, (b) we show the existence of subgame-perfect equilibria such that whenever there is only one active player at some stage, the player can play pure strategy as part of the equilibrium, and (c) a new existence result is obtained for stochastic games.

5.1 *Measurable dynamic games with almost perfect information*

We follow the setting and notation in Section 4.1 as closely as possible, and describe only the changes we need to make on the model. In Section 4.1, we assume that the relevant model parameters (action correspondences, Nature's move, and payoff functions) are continuous in both actions and states. Here, we work with the class of games with sectionally continuous model parameters in the following sense. Suppose that Y_1, Y_2, and Y_3 are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$. Denote $Z(y_1) = \{y_2 \in Y_2 \colon (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f \colon Z \to Y_3$ is said to be sectionally continuous on Y_2 if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for all y_1 with $Z(y_1) \neq \varnothing$. Similarly, one can define the sectional upper hemicontinuity for a correspondence.

Compared with continuous dynamic games with almost perfect information, the changes we need to make to describe measurable dynamic games are as follows.

- For any $t \geq 1$ and $i \in I$, A_{ti} is sectionally continuous on X^{t-1}.[29]
- For any $t \geq 1$, f_{t0} is sectionally continuous on X^{t-1}.
- For each $i \in I$, the payoff function u_i is sectionally continuous on X^{∞}.

For each $t \geq 0$, suppose that λ_t is a Borel probability measure on S_t and λ_t is atomless for $t \geq 1$. Let $\lambda^t = \bigotimes_{0 \leq k \leq t} \lambda_k$ for $t \geq 0$. We assume the following condition on the state transitions.

Assumption 3 (Atomless Reference Measure). *A dynamic game is said to satisfy the atomless reference measure (ARM) condition if for each $t \geq 1$, the following statements hold.*

(i) The probability $f_{t0}(\cdot|h_{t-1})$ is absolutely continuous with respect to λ_t on S_t with the Radon–Nikodym derivative $\varphi_{t0}(h_{t-1}, s_t)$ for all $h_{t-1} \in H_{t-1}$.[30]

[29]Note that a history mixes the multiple components of states and actions in different periods. As noted in footnote 15, one can also view a history h_{t-1} as an element in $X^{t-1} \times S^{t-1}$ by abusing the notation.

[30]It is common to have a reference measure when one considers a game with uncountable states. For example, if S_t is a convex subset of $\mathbb{R}^l$, then the uniform distribution on the convex set is a natural reference measure. In particular, the condition that the state transitions are absolutely continuous with respect to a reference measure is widely adopted in the literature on stochastic games; see, for example, Nowak (1985), Nowak and Raghavan (1992), Duffie et al. (1994), and He and Sun (2017).

(ii) The mapping φ_{t0} is Borel measurable and sectionally continuous on X^{t-1}, and integrably bounded in the sense that there is a λ_t-integrable function $\phi_t\colon S_t \to \mathbb{R}_+$ such that $\varphi_{t0}(h_{t-1}, s_t) \le \phi_t(s_t)$ for any $h_{t-1} \in H_{t-1}$ and $s_t \in S_t$.

When one considers a dynamic game with infinite horizon, the following "continuity at infinity" condition in (1) is standard.[31] This condition means that the actions and states in the far future do not matter that much for any player's payoff. In particular, all discounted repeated games or stochastic games satisfy this condition.

For any $T \ge 1$, let

$$w^T = \sup_{\substack{i\in I\\ (x,s)\in H_\infty\\ (\overline{x},\overline{s})\in H_\infty\\ x^{T-1}=\overline{x}^{T-1}\\ s^{T-1}=\overline{s}^{T-1}}} \left|u_i(x,s) - u_i(\overline{x},\overline{s})\right|. \tag{1}$$

Assumption 4 (Continuity at Infinity). *A dynamic game is said to be continuous at infinity if $w^T \to 0$ as $T \to \infty$.*

We modify the notion of subgame-perfect equilibrium slightly. In particular, when the state space is uncountable and has a reference measure, it is natural to consider the optimality for almost all subhistories in the probabilistic sense:[32] a property is said to hold for λ^t-almost all $h_t = (x^t, s^t) \in H_t$ if it is satisfied for λ^t-almost all $s^t \in S^t$ and all $x^t \in H_t(s^t)$.

Definition 3 (SPE). A subgame-perfect equilibrium is a strategy profile f such that for all $i \in I$, $t \ge 0$, and λ^t-almost all $h_t \in H_t$, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy.

The theorem below shows the existence of a subgame-perfect equilibrium under the conditions of ARM and continuity at infinity. Recall that $E_t(h_{t-1})$ is the set of all subgame-perfect equilibrium payoffs in the subgame h_{t-1}. The theorem also shows the compactness and upper hemicontinuity properties of the correspondence E_t. In particular, we work with the upper hemicontinuity property also in the probabilistic sense. Suppose that Y_1, Y_2, and Y_3 are all Polish spaces, and $Z \subseteq Y_1 \times Y_2$ and η is a Borel probability measure on Y_1. Denote $Z(y_1) = \{y_2 \in Y_2\colon (y_1, y_2) \in Z\}$ for any $y_1 \in Y_1$. A function (resp. correspondence) $f\colon Z \to Y_3$ is said to be essentially sectionally continuous on Y_2 if $f(y_1, \cdot)$ is continuous on $Z(y_1)$ for η-almost all y_1. Similarly, one can define the essential sectional upper hemicontinuity for a correspondence.

Theorem 3. *If a dynamic game with almost perfect information satisfies the ARM condition and is continuous at infinity, then it possesses a subgame-perfect equilibrium. In addition, E_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1}.*

[31] See, for example, Fudenberg and Levine (1983).

[32] See, for example, Abreu et al. (1990) and footnote 4 therein.

5.2 *Measurable dynamic games with perfect information*

In this subsection, we consider dynamic games with perfect information (with or without Nature). We follow the setting and notation in Section 4.2, and make the same changes as those in Section 5.1. In particular, the continuity requirement in the state variables is dropped.

In dynamic games with perfect information where players move sequentially, we show the existence of pure-strategy subgame-perfect equilibria. The ARM condition is imposed when Nature moves and is dropped in those periods with one active player from the set I.[33]

THEOREM 4. *If a dynamic game with perfect information satisfies the ARM condition and is continuous at infinity, then it possesses a pure-strategy subgame-perfect equilibrium. In addition, E_t is nonempty and compact valued, and essentially sectionally upper hemicontinuous on X^{t-1} for any $t \geq 1$.*

5.3 *A road map for proving Theorems 3 and 4*

The logic for the proofs of Theorems 3 and 4 is similar to that for the proofs of Theorems 1 and 2. The existence results are also established in three steps. However, new subtle difficulties arise. In this subsection, we summarize the main changes for proving Theorems 3 and 4. For simplicity, we omit the repetitive descriptions and adopt the same notation as in Section 4.3.

In the first (backward induction) step, following the same argument as in Section 4.3, one can construct the correspondence P_t. Recall that the key role of the atomless transition condition is to guarantee that P_t is convex-valued and upper hemicontinuous. With the condition of atomless reference measure, one is still able to show that P_t is convex-valued. However, though the correspondence P_t remains upper hemicontinuous in actions, it is only measurable with respect to the states. As a result, the existence result in Simon and Zame (1990) is not readily applicable. We extend their existence result by allowing the payoff correspondence to be upper hemicontinuous in actions, but measurable in states. The key to this extension is to approximate the measurable correspondence by continuous correspondences based on Lusin's theorem (see Lemma 3).

In the forward induction step, an important observation is that the set of histories H_{t-1} at stage t can be divided into countably many Borel subsets $\{H_{t-1}^m\}_{m\geq 0}$ with desirable properties. In particular, we have the following relationships:

- We have $H_{t-1} = \bigcup_{m\geq 0} H_{t-1}^m$ and $\frac{\lambda^{t-1}(\bigcup_{m\geq 1} \mathrm{proj}_{S^{t-1}}(H_{t-1}^m))}{\lambda^{t-1}(\mathrm{proj}_{S^{t-1}}(H_{t-1}))} = 1$, where $\mathrm{proj}_{S^{t-1}}(H_{t-1}^m)$ and $\mathrm{proj}_{S^{t-1}}(H_{t-1})$ are projections of H_{t-1}^m and H_{t-1} on S^{t-1}.
- For $m \geq 1$, H_{t-1}^m is compact, Φ_t is upper hemicontinuous on H_{t-1}^m, and P_t is upper hemicontinuous on

$$\big\{(h_{t-1}, x_t)\colon h_{t-1} \in H_{t-1}^m, x_t \in A_t(h_{t-1})\big\}.$$

[33]As noted in Remark 4, Theorem 4 can be generalized to the case when the state transitions either satisfy the ARM condition or have the support inside a fixed finite set irrespective of the history at a particular stage.

Note that within each compact subset H_{t-1}^m for $m \geq 1$, the correspondences Φ_t and P_t are well behaved. One can apply the forward induction argument from the proof for continuous dynamic games to each H_{t-1}^m, which enables us to obtain a strategy defined on this subset H_{t-1}^m. The forward induction step for measurable dynamic games is then completed by combining the equilibrium strategies obtained on H_{t-1}^m, $m \geq 0$ (subject to slight modifications).

The last step (extending the finite-horizon setting to the infinite-horizon setting) follows a logic similar to that explained in Step 3 in Section 4.3. The main challenge is to handle various subtle measurability issues due to the lack of continuity in the state variables. As described in Section 4.3, the idea of this step is to show the upper hemicontinuity of equilibrium payoff correspondences in an infinite horizon. In the case of continuous dynamic games, this property is shown, based on a few technical lemmas on upper hemicontinuous correspondences. For the class of measurable dynamic games as considered here, we need to extend those technical lemmas to the more difficult case of measurable correspondences.

Appendix A

In Appendix A.1, we present several lemmas as the mathematical preparations for proving Theorems 1 and 2, and Proposition 1. Since correspondences are used extensively in the proofs, we collect, for the convenience of the reader, several known results on various properties of correspondences.[34] One can skip Appendix A.1 first and go to the proofs in Appendices A.2–A.4 directly, and then refer to those technical lemmas in Appendix A.1 whenever necessary.

The proof of Theorem 1 is provided in Appendix A.2. In Appendices A.3 and A.4, we give the proofs of Theorem 2 and Proposition 1, respectively. We describe only the necessary changes in comparison with the proofs presented in Appendices A.2.1–A.2.3.

A.1 *Technical preparations*

Let $(S, \mathcal{S})$ be a measurable space and let X be a topological space with its Borel σ-algebra $\mathcal{B}(X)$. A correspondence Ψ from S to X is a function from S to the space of all subsets of X. A mapping ψ is said to be a selection of Ψ if $\psi(s) \in \Psi(s)$ for any $s \in S$. The upper inverse Ψ^u of a subset $A \subseteq X$ is

$$\Psi^u(A) = \{s \in S \colon \Psi(s) \subseteq A\}.$$

The lower inverse Ψ^l of a subset $A \subseteq X$ is

$$\Psi^l(A) = \{s \in S \colon \Psi(s) \cap A \neq \varnothing\}.$$

The correspondence Ψ is

(a) weakly measurable if $\Psi^l(O) \in \mathcal{S}$ for each open subset $O \subseteq X$

[34]These technical lemmas are stated in a general form so that they can still be used in the proofs in Appendix B for the case of measurable dynamic games.

(b) measurable if $\Psi^l(K) \in \mathcal{S}$ for each closed subset $K \subseteq X$.

The graph of Ψ is denoted by $\mathrm{Gr}(\Psi) = \{(s, x) \in S \times X : s \in S, x \in \Psi(s)\}$. The correspondence Ψ is said to have a measurable graph if $\mathrm{Gr}(\Psi) \in \mathcal{S} \otimes \mathcal{B}(X)$.

If S is a topological space, then Ψ is

(i) upper hemicontinuous if $\Psi^u(O)$ is open for each open subset $O \subseteq X$

(ii) lower hemicontinuous if $\Psi^l(O)$ is open for each open subset $O \subseteq X$

(iii) continuous if it is both upper hemicontinuous and lower hemicontinuous.

The following two lemmas present some basic measurability and continuity properties for correspondences.

Lemma 1. *Let $(S, \mathcal{S})$ be a measurable space, X a Polish space endowed with the Borel σ-algebra $\mathcal{B}(X)$, and $\mathcal{K}$ the space of nonempty compact subsets of X endowed with its Hausdorff metric topology. Suppose that $\Psi : S \to X$ is a nonempty and closed valued correspondence.*

(i) If Ψ is weakly measurable, then it has a measurable graph.

(ii) If Ψ is compact-valued, then the following statements are equivalent.

(a) The correspondence Ψ is weakly measurable.

(b) The correspondence Ψ is measurable.

(c) The function $f : S \to \mathcal{K}$, defined by $f(s) = \Psi(s)$, is Borel measurable.

(iii) Suppose that S is a topological space. If Ψ is compact-valued, then the function $f : S \to \mathcal{K}$, defined by $f(s) = \Psi(s)$, is continuous if and only if the correspondence Ψ is continuous.

(iv) Suppose that $(S, \mathcal{S}, \lambda)$ is a complete probability space. Then Ψ is $\mathcal{S}$-measurable if and only if it has a measurable graph.

(v) For a correspondence $\Psi : S \to X$ between two Polish spaces, the following statements are equivalent.

(a) The correspondence Ψ is upper hemicontinuous at a point $s \in S$ and $\Psi(s)$ is compact.

(b) If a sequence (s_n, x_n) in the graph of Ψ satisfies $s_n \to s$, then the sequence $\{x_n\}$ has a limit in $\Psi(s)$.

(vi) For a correspondence $\Psi : S \to X$ between two Polish spaces, the following statements are equivalent.

(a) The correspondence Ψ is lower hemicontinuous at a point $s \in S$.

(b) If $s_n \to s$, then for each $x \in \Psi(s)$, there exist a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ and elements $x_k \in \Psi(s_{n_k})$ for each k such that $x_k \to x$.

(vii) Given correspondences $F\colon X \to Y$ and $G\colon Y \to Z$, the composition F and G is defined by

$$G\big(F(x)\big) = \bigcup_{y \in F(x)} G(y).$$

The composition of upper hemicontinuous correspondences is upper hemicontinuous. The composition of lower hemicontinuous correspondences is lower hemicontinuous.

Properties (i), (ii), (iii), (v), (vi), and (vii) are, respectively, Theorems 18.6, 18.10, 17.15, 17.20, 17.21, and 17.23 of Aliprantis and Border (2006). Property (iv) follows from Proposition 4 on page 61 of Hildenbrand (1974).

Lemma 2. *(i) A correspondence Ψ from a measurable space $(S, \mathcal{S})$ into a topological space X is weakly measurable if and only if its closure correspondence $\overline{\Psi}$ is weakly measurable, where $\overline{\Psi}(s)$ is the closure of the set $\Psi(s)$ in X for each $s \in S$.*

(ii) For a sequence $\{\Psi_m\}$ of correspondences from a measurable space $(S, \mathcal{S})$ into a Polish space, the union correspondence $\Psi(s) = \bigcup_{m \geq 1} \Psi_m(s)$ is weakly measurable if each Ψ_m is weakly measurable. If each Ψ_m is weakly measurable and compact-valued, then the intersection correspondence $\Phi(s) = \bigcap_{m \geq 1} \Psi_m(s)$ is weakly measurable.

(iii) A weakly measurable, nonempty and closed valued correspondence from a measurable space into a Polish space admits a measurable selection.

(iv) A correspondence with a closed graph between compact metric spaces is measurable.

(v) A nonempty and compact valued correspondence Ψ from a measurable space $(S, \mathcal{S})$ into a Polish space is weakly measurable if and only if there exists a sequence $\{\psi_1, \psi_2 \ldots\}$ of measurable selections of Ψ such that $\Psi(s) = \overline{\Phi}(s)$, where $\Phi(s) = \{\psi_1(s), \psi_2(s), \ldots\}$ for each $s \in S$.

(vi) The image of a compact set under a compact-valued upper hemicontinuous correspondence is compact.[35] *If the domain is compact, then the graph of a compact-valued upper hemicontinuous correspondence is compact.*

(vii) The intersection of a correspondence with closed graph and an upper hemicontinuous compact-valued correspondence is upper hemicontinuous.

(viii) If the correspondence $\Psi\colon S \to \mathbb{R}^l$ is compact-valued and upper hemicontinuous, then the convex hull of Ψ is also compact-valued and upper hemicontinuous.

[35]Given a correspondence $F\colon X \to Y$ and a subset A of X, the image of A under F is defined to be the set $\bigcup_{x \in A} F(x)$.

Properties (i)–(vii) are, respectively, Lemmas 18.3 and 18.4, Theorems 18.13 and 18.20, Corollary 18.15, Lemma 17.8, and Theorem 17.25 in Aliprantis and Border (2006). Property (viii) is Proposition 6 on page 26 of Hildenbrand (1974).

Parts (i) and (ii) of the following lemma are the standard Lusin theorem and Michael's continuous selection theorem, while the other two parts are about properties that involve mixture of measurability and continuity of correspondences.

Lemma 3. *(i) Lusin's theorem: Suppose that S is a Borel subset of a Polish space, λ is a Borel probability measure on S, and $\mathcal{S}$ is the completion of $\mathcal{B}(S)$ under λ. Let X be a Polish space. If f is an $\mathcal{S}$-measurable mapping from S to X, then for any $\epsilon > 0$, there exists a compact subset $S_1 \subseteq S$ with $\lambda(S \setminus S_1) < \epsilon$ such that the restriction of f to S_1 is continuous.*

(ii) Michael's continuous selection theorem: Let S be a metrizable space, and X be a complete metrizable closed subset of some locally convex space. Suppose that $F\colon S \to X$ is a lower hemicontinuous, nonempty, convex and closed valued correspondence. Then there exists a continuous mapping $f\colon S \to X$ such that $f(s) \in F(s)$ for all $s \in S$.

(iii) Let $(S, \mathcal{S}, \lambda)$ be a finite measure space, X be a Polish space, and Y be a locally convex linear topological space. Let $F\colon S \to X$ be a closed-valued correspondence such that $\mathrm{Gr}(F) \in \mathcal{S} \otimes \mathcal{B}(X)$, and $f\colon \mathrm{Gr}(F) \to Y$ be a measurable function that is sectionally continuous on X. Then there exists a measurable function $f'\colon S \times X \to Y$ such that (a) f' is sectionally continuous on X, (b) for λ-almost all $s \in S$, $f'(s, x) = f(s, x)$ for all $x \in F(s)$ and $f'(s, X) \subseteq \mathrm{co}\, f(s, F(s))$.[36]

(iv) Let $(S, \mathcal{S})$ be a measurable space, and X and Y be Polish spaces. Let $\Psi\colon S \times X \to \mathcal{M}(Y)$ be an $\mathcal{S} \otimes \mathcal{B}(X)$-measurable, nonempty, convex and compact valued correspondence that is sectionally continuous on X, where the compactness and continuity are with respect to the weak* topology on $\mathcal{M}(Y)$. Then there exists an $\mathcal{S} \otimes \mathcal{B}(X)$-measurable selection ψ of Ψ that is sectionally continuous on X.

Lusin's theorem is Theorem 7.1.13 in Bogachev (2007). Michael's continuous selection theorem can be found in Michael (1966) and the last paragraph on page 228 of Bogachev (2007). Property (iii) is Theorem 2.7 in Brown and Schreiber (1989). Property (iv) follows from Theorem 1 and the main lemma of Kucia (1998).

The following lemma presents the convexity, compactness, and continuity properties on integration of correspondences.

Lemma 4. *Let $(S, \mathcal{S}, \lambda)$ be an atomless probability space, X be a Polish space, and F be a correspondence from S to $\mathbb{R}^l$. Denote*

$$\int_S F(s)\lambda(ds) = \left\{ \int_S f(s)\lambda(ds) \colon f \text{ is an integrable selection of } F \text{ on } S \right\}.$$

[36]For any set A in a linear topological space, co A denotes the convex hull of A.

(i) *If F is measurable, nonempty and closed valued, and λ-integrably bounded by some integrable function $\psi\colon S \to \mathbb{R}_+$ in the sense that for λ-almost all $s \in S$, $\|y\| \le \psi(s)$ for any $y \in F(s)$, then $\int_S F(s)\lambda(ds)$ is nonempty, convex, and compact, and*

$$\int_S F(s)\lambda(ds) = \int_S \operatorname{co} F(s)\lambda(ds).$$

(ii) *If G is a measurable, nonempty and closed valued correspondence from $S \times X \to \mathbb{R}^l$ such that (a) $G(s,\cdot)$ is upper (resp. lower) hemicontinuous on X for all $s \in S$ and (b) G is λ-integrably bounded by some integrable function $\psi\colon S \to \mathbb{R}_+$ in the sense that for λ-almost all $s \in S$, $\|y\| \le \psi(s)$ for any $x \in X$ and $y \in G(s,x)$, then $\int_S G(s,x)\lambda(ds)$ is upper (resp. lower) hemicontinuous on X.*

For the proof, see Theorems 2, 3, and 4, Propositions 7 and 8, and Problem 6 in Section D.II.4 of Hildenbrand (1974).

The following result proves a measurable version of Lyapunov's theorem, which is taken from Mertens (2003). Let $(S,\mathcal{S})$ and $(X,\mathcal{X})$ be measurable spaces. A transition probability from S to X is a mapping f from S to the space $\mathcal{M}(X)$ of probability measures on $(X,\mathcal{X})$ such that $f(B|\cdot) : s \to f(B|s)$ is $\mathcal{S}$-measurable for each $B \in \mathcal{X}$.

Lemma 5. *Let $f(\cdot|s)$ be a transition probability from a measurable space $(S,\mathcal{S})$ to another measurable space $(X,\mathcal{X})$ ($\mathcal{X}$ is separable).*[37] *Let Q be a measurable, nonempty and compact valued correspondence from $S \times X$ to $\mathbb{R}^l$, which is f-integrable in the sense that for any measurable selection q of Q, $q(s,\cdot)$ is $f(\cdot|s)$-absolutely integrable for any $s \in S$. Let $\int Q\,df$ be the correspondence from S to subsets of $\mathbb{R}^l$ defined by*

$$M(s) = \left(\int Q\,df\right)(s) = \left\{\int_X q(s,x)f(dx|s) \colon q \text{ is a measurable selection of } Q\right\}.$$

Denote the graph of M by J. Let $\mathcal{J}$ be the restriction of the product σ-algebra $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}^l)$ to J. Then the following statements hold.

(i) *The term M is a measurable, nonempty and compact valued correspondence.*

(ii) *There exists a measurable, $\mathbb{R}^l$-valued function g on $(X \times J, \mathcal{X} \otimes \mathcal{J})$ such that $g(x,e,s) \in Q(x,s)$ and $e = \int_X g(x,e,s)f(dx|s)$.*

The proof of Lemma 6 in Reny and Robson (2002) leads to the following result.

Lemma 6. *Suppose that H and X are Polish spaces. Let $P\colon H \times X \to \mathbb{R}^n$ be a measurable, nonempty and compact valued correspondence, and let the mappings $f\colon H \to \mathcal{M}(X)$ and $\mu\colon H \to \triangle(X)$ be measurable, where $\triangle(X)$ is the set of all finite Borel measures on X. In addition, suppose that $\mu(\cdot|h) = p(h,\cdot) \circ f(\cdot|h)$ such that $p(h,\cdot)$ is a measurable selection of $P(h,\cdot)$ for each h.*[38] *Then there exists a jointly Borel measurable selection g of P such that $\mu(\cdot|h) = g(h,\cdot) \circ f(\cdot|h)$; that is, $g(h,x) = p(h,x)$ for $f(\cdot|h)$-almost all x.*

[37] A σ-algebra is said to be separable if it is generated by a countable collection of sets.

[38] The finite measure $\mu(\cdot|h) = p(h,\cdot) \circ f(\cdot|h)$ if $\mu(E|h) = \int_E p(h,x)f(dx|h)$ for any Borel set E.

Suppose that $(S_1, \mathcal{S}_1)$ is a measurable space, S_2 is a Polish space endowed with the Borel σ-algebra, and $S = S_1 \times S_2$, which is endowed with the product σ-algebra $\mathcal{S}$. Let D be an $\mathcal{S}$-measurable subset of S such that $D(s_1)$ is compact for any $s_1 \in S_1$. The σ-algebra $\mathcal{D}$ is the restriction of $\mathcal{S}$ on D. Let X be a Polish space, and let A be a $\mathcal{D}$-measurable, nonempty and closed valued correspondence from D to X that is sectionally continuous on S_2. The following lemma considers the property of upper hemicontinuity for the correspondence M as defined in Lemma 5.

Lemma 7. *Let $f(\cdot|s)$ be a transition probability from $(D, \mathcal{D})$ to $\mathcal{M}(X)$ such that $f(A(s)|s) = 1$ for any $s \in D$, which is sectionally continuous on S_2. Let G be a bounded, measurable, nonempty, convex and compact valued correspondence from $\mathrm{Gr}(A)$ to $\mathbb{R}^l$, which is sectionally upper hemicontinuous on $S_2 \times X$. Let $\int G\,df$ be the correspondence from D to subsets of $\mathbb{R}^l$ defined by*

$$M(s) = \left(\int G\,df\right)(s) = \left\{\int_X g(s,x) f(dx|s) \colon g \text{ is a measurable selection of } G\right\}.$$

Then M is $\mathcal{S}$-measurable, nonempty and compact valued, and sectionally upper hemicontinuous on S_2.

Proof. Define a correspondence $\tilde{G} \colon S \times X \to \mathbb{R}^l$ as

$$\tilde{G} = \begin{cases} G(s,x), & \text{if } (s,x) \in \mathrm{Gr}(A), \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then $M(s) = (\int \tilde{G}\,df)(s) = (\int G\,df)(s)$. The measurability, nonemptiness, and compactness follow from Lemma 5. Given $s_1 \in S_1$ such that (a) $D(s_1) \neq \varnothing$, (b) $f(s_1, \cdot)$, and $G(s_1, \cdot, \cdot)$ is upper hemicontinuous. The upper hemicontinuity of $M(s_1, \cdot)$ follows from Lemma 2 in Simon and Zame (1990) and Lemma 4 in Reny and Robson (2002). □

From now on, whenever we work with mappings taking values in the space $\mathcal{M}(X)$ of Borel probability measures on some separable metric space X, the relevant continuity or convergence is assumed to be in terms of the topology of weak convergence of measures on $\mathcal{M}(X)$ unless otherwise noted.

In the following lemma, we state some properties for transition correspondences.

Lemma 8. *Suppose that Y and Z are Polish spaces. Let G be a measurable, nonempty, convex and compact valued correspondence from Y to $\mathcal{M}(Z)$. Define a correspondence G' from $\mathcal{M}(Y)$ to $\mathcal{M}(Z)$ as*

$$G'(\nu) = \left\{\int_Y g(y)\nu(dy) \colon g \text{ is a Borel measurable selection of } G\right\}.$$ [39]

[39] The integral $\int_Y g(y)\nu(dy)$ defines a Borel probability measure τ on Z such that for any Borel set C in Z, $\tau(C) = \int_Y g(y)(C)\nu(dy)$. The measure τ is also equal to the Gelfand integral of g with respect to the measure ν on Y, when $\mathcal{M}(Z)$ is viewed as a set in the dual space of the space of bounded continuous functions on Z; see Definition 11.49 of Aliprantis and Border (2006).

(i) The correspondence G' is measurable, nonempty, convex and compact valued.

(ii) The correspondence G is upper hemicontinuous if and only if G' is upper hemicontinuous. In addition, if G is continuous, then G' is continuous.

Proof. Part (i) is Lemma 19.29 of Aliprantis and Border (2006). By Theorem 19.30 therein, G is upper hemicontinuous if and only if G' is upper hemicontinuous. We need to show that G' is lower hemicontinuous if G is lower hemicontinuous.

Let Z be endowed with a totally bounded metric, and $U(Z)$ the space of bounded, real-valued, and uniformly continuous functions on Z endowed with the supremum norm, which is obviously separable. Pick a countable set $\{f_m\}_{m\geq 1} \subseteq U(Z)$ such that $\{f_m\}$ is dense in the unit ball of $U(Z)$. It follows from Theorem 15.2 of Aliprantis and Border (2006) that the weak* topology of $\mathcal{M}(Z)$ is metrizable by the metric d_z, where

$$d_z(\mu_1, \mu_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \int_Z f_m(z)\mu_1(dz) - \int_Z f_m(z)\mu_2(dz) \right|$$

for each pair of $\mu_1, \mu_2 \in \mathcal{M}(Z)$.

Suppose that $\{\nu_j\}_{j\geq 0}$ is a sequence in $\mathcal{M}(Y)$ such that $\nu_j \to \nu_0$ as $j \to \infty$. Pick an arbitrary point $\mu_0 \in G'(\nu_0)$. By the definition of G', there exists a Borel measurable selection g of G such that $\mu_0 = \int_Y g(y)\nu_0(dy)$.

For each $k \geq 1$, by Lemma 3 (Lusin's theorem), there exists a compact subset $D_k \subseteq Y$ such that g is continuous on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$. Define a correspondence $G_k : Y \to \mathcal{M}(Z)$ as

$$G_k(y) = \begin{cases} \{g(y)\}, & y \in D_k \\ G(y), & y \in Y \setminus D_k. \end{cases}$$

Then G_k is nonempty, convex and compact valued, and lower hemicontinuous. By Theorem 3.22 in Aliprantis and Border (2006), Y is paracompact. Then by Lemma 3 (Michael's selection theorem), it has a continuous selection g_k.

For each k, since $\nu_j \to \nu_0$ and g_k is continuous, $\int_Y g_k(y)\nu_j(dy) \to \int_Y g_k(y)\nu_0(dy)$ in the sense that for any $m \geq 1$,

$$\int_Y \int_Z f_m(z) g_k(dz|y)\nu_j(dy) \to \int_Y \int_Z f_m(z) g_k(dz|y)\nu_0(dy).$$

Thus, there exists a point ν_{j_k} such that $\{j_k\}$ is an increasing sequence and

$$d_z\left(\int_Y g_k(y)\nu_{j_k}(dy), \int_Y g_k(y)\nu_0(dy)\right) < \frac{1}{3k}.$$

In addition, since g_k coincides with g on D_k and $\nu_0(Y \setminus D_k) < \frac{1}{3k}$,

$$d_z\left(\int_Y g_k(y)\nu_0(dy), \int_Y g(y)\nu_0(dy)\right) < \frac{2}{3k}.$$

Thus,

$$d_z\left(\int_Y g_k(y)\nu_{j_k}(dy), \int_Y g(y)\nu_0(dy)\right) < \frac{1}{k}.$$

Let $\mu_{j_k} = \int_Y g_k(y)\nu_{j_k}(dy)$ for each k. Then $\mu_{j_k} \in G'(\nu_{j_k})$ and $\mu_{j_k} \to \mu_0$ as $k \to \infty$. By Lemma 1, G' is lower hemicontinuous. □

The next lemma presents some properties for the composition of two transition correspondences in terms of the product of transition probabilities.

Lemma 9. *Let X, Y, and Z be Polish spaces, and G a measurable, nonempty and compact valued correspondence from X to $\mathcal{M}(Y)$. Suppose that F is a measurable, nonempty, convex and compact valued correspondence from $X \times Y$ to $\mathcal{M}(Z)$. Define a correspondence Π from X to $\mathcal{M}(Y \times Z)$ as*

$$\Pi(x) = \big\{g(x) \diamond f(x) \colon g \text{ is a Borel measurable selection of } G, \\ f \text{ is a Borel measurable selection of } F\big\}.$$

(i) If F is sectionally continuous on Y, then Π is a measurable, nonempty and compact valued correspondence.

(ii) If there exists a function g from X to $\mathcal{M}(Y)$ such that $G(x) = \{g(x)\}$ for any $x \in X$, then Π is a measurable, nonempty and compact valued correspondence.

(iii) If both G and F are continuous correspondences, then Π is a nonempty and compact valued, and continuous correspondence.[40]

(iv) If $G(x) \equiv \{\lambda\}$ for some fixed Borel probability measure $\lambda \in \mathcal{M}(Y)$ and F is sectionally continuous on X, then Π is a continuous, nonempty and compact valued correspondence.

Proof. (i) Define three correspondences $\tilde{F}\colon X \times Y \to \mathcal{M}(Y \times Z)$, $\hat{F}\colon \mathcal{M}(X \times Y) \to \mathcal{M}(Y \times Z)$, and $\check{F}\colon X \times \mathcal{M}(Y) \to \mathcal{M}(Y \times Z)$ as

$$\tilde{F}(x, y) = \big\{\delta_y \otimes \mu \colon \mu \in F(x, y)\big\},$$

$$\hat{F}(\tau) = \left\{\int_{X\times Y} f(x, y)\tau\big(d(x, y)\big) \colon f \text{ is a Borel measurable selection of } \tilde{F}\right\},$$

$$\check{F}(x, \mu) = \hat{F}(\delta_x \otimes \mu).$$

Since F is measurable, nonempty, convex and compact valued, $\tilde{F}$ is measurable, nonempty, convex and compact valued. By Lemma 8, the correspondence $\hat{F}$ is measurable, nonempty, convex and compact valued, and $\check{F}(x, \cdot)$ is continuous on $\mathcal{M}(Y)$ for any $x \in X$.

[40] In Lemma 29 of Harris et al. (1995), they showed that Π is upper hemicontinuous if both G and F are upper hemicontinuous.

Since G is measurable and compact-valued, there exists a sequence of Borel measurable selections $\{g_k\}_{k\geq 1}$ of G such that $G(x) = \overline{\{g_1(x), g_2(x), \ldots\}}$ for any $x \in X$ by Lemma 2(v). For each $k \geq 1$, define a correspondence Π^k from X to $\mathcal{M}(Y \times Z)$ by letting $\Pi^k(x) = \check{F}(x, g_k(x)) = \hat{F}(\delta_x \otimes g_k(x))$. Then Π^k is measurable, nonempty, convex and compact valued.

Fix any $x \in X$. It is clear that $\Pi(x) = \check{F}(x, G(x))$ is nonempty-valued. Since $G(x)$ is compact, and $\check{F}(x, \cdot)$ is compact-valued and continuous, $\Pi(x)$ is compact by Lemma 2. Thus, the closure $\overline{\bigcup_{k\geq 1} \Pi^k(x)}$ of $\bigcup_{k\geq 1} \Pi^k(x)$ is a subset of $\Pi(x)$.

Fix any $x \in X$ and $\tau \in \Pi(x)$. There exists a point $\nu \in G(x)$ such that $\tau \in \check{F}(x, \nu)$. Since $\{g_k(x)\}_{k\geq 1}$ is dense in $G(x)$, it has a subsequence $\{g_{k_m}(x)\}$ such that $g_{k_m}(x) \to \nu$. As $\check{F}(x, \cdot)$ is continuous, $\check{F}(x, g_{k_m}(x)) \to \check{F}(x, \nu)$. That is,

$$\tau \in \overline{\bigcup_{k\geq 1} \check{F}\big(x, g_k(x)\big)} = \overline{\bigcup_{k\geq 1} \Pi^k(x)}.$$

Therefore, $\overline{\bigcup_{k\geq 1} \Pi^k(x)} = \Pi(x)$ for any $x \in X$. Lemma 2(i) and (ii) imply that Π is measurable.

(ii) As in (i), the correspondence $\hat{F}$ is measurable, nonempty, convex and compact valued. If $G = \{g\}$ for some measurable function g, then $\Pi(x) = \hat{F}(\delta_x \otimes g(x))$, which is measurable, nonempty and compact valued.

(iii) We continue to work with the two correspondences $\tilde{F}\colon X \times Y \to \mathcal{M}(Y \times Z)$ and $\hat{F}\colon \mathcal{M}(X \times Y) \to \mathcal{M}(Y \times Z)$ as in part (i). By the condition on F, it is obvious that the correspondence $\tilde{F}$ is continuous, nonempty, convex and compact valued. Lemma 8 implies the properties for the correspondence $\hat{F}$. Define a correspondence $\hat{G}\colon X \to \mathcal{M}(X \times Y)$ as $\hat{G}(x) = \delta_x \otimes G(x)$.[41] Since $\hat{G}$ and $\hat{F}$ are both nonempty-valued, $\Pi(x) = \hat{F}(\hat{G}(x))$ is nonempty. As $\hat{G}$ is compact-valued and $\hat{F}$ is continuous, Π is compact-valued by Lemma 2. As $\hat{G}$ and $\hat{F}$ are both continuous, Π is continuous by Lemma 1(vii).

(iv) Let $Y' = Y$ and define a correspondence $\check{F}\colon X \times Y \to \mathcal{M}(Y' \times Z)$ as

$$\check{F}(x, y) = \delta_y \otimes F(x, y) = \big\{\delta_y \otimes \mu\colon \mu \in F(x, y)\big\}.$$

Then $\check{F}$ is also measurable, nonempty, convex and compact valued, and sectionally upper hemicontinuous on X.

Let d be a totally bounded metric on $Y' \times Z$, and $U(Y' \times Z)$ be the space of bounded, real-valued, and uniformly continuous functions on $Y' \times Z$ endowed with the supremum norm, which is obviously separable. It follows from Theorem 15.2 of Aliprantis and Border (2006) that the space of Borel probability measures on $Y' \times Z$ with the topology of weak convergence of measures can be viewed as a subspace of the dual space of $U(Y' \times Z)$ with the weak* topology. By Corollary 18.37 of Aliprantis and Border (2006), $\Pi(x) = \int_Y \check{F}(x, y)\lambda(dy)$ is nonempty, convex, and compact for any $x \in X$.[42]

[41] Given a finite measure ν on X and a set D of finite measures on Y, $\nu \otimes D$ denotes the set of finite measures $\{\nu \otimes \mu : \mu \in D\}$.

[42] Note that the integral $\int_Y \check{F}(x, y)\lambda(dy)$ can be viewed as the Gelfand integral of the correspondence in the dual space of $U(Y' \times Z)$; see Definition 18.36 of Aliprantis and Border (2006).

Now we show the upper hemicontinuity. If $x_n \to x_0$ and $\mu_n \in \Pi(x_n)$, we need to prove that there exists some $\mu_0 \in \Pi(x_0)$ such that a subsequence of $\{\mu_n\}$ weakly converges to μ_0. Suppose that for $n \geq 1$, f_n is a Borel measurable selection of $F(x_n, \cdot)$ such that $\mu_n = \lambda \diamond f_n$.

Fix any $y \in Y$ and let $J(y) = \overline{\text{co}}\{f_n(y) \otimes \delta_{x_n}\}_{n\geq 1}$, which is the closure of the convex hull of $\{f_n(y) \otimes \delta_{x_n}\}_{n\geq 1}$. It is obvious that $J(y)$ is nonempty and convex. It is also clear that $J(y)$ is the closure of the countable set

$$\left\{\sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i} : n \geq 1, \alpha_i \in \mathbb{Q}_+, i = 1, \ldots, n, \sum_{i=1}^{n} \alpha_i = 1\right\},$$

where $\mathbb{Q}_+$ is the set of nonnegative rational numbers. Let $F'(x) = \{\mu \otimes \delta_x : \mu \in F(x, y)\}$ for any $x \in X$. Then F' is continuous on X. Since $\{x_n : n \geq 0\}$ is a compact set, Lemma 2(vi) implies that $\bigcup_{n\geq 0} F'(x_n)$ is compact. Hence, $\{f_n(y) \otimes \delta_{x_n}\}_{n\geq 1}$ is relatively compact. By Theorem 5.22 of Aliprantis and Border (2006), $\{f_n(y) \otimes \delta_{x_n}\}_{n\geq 1}$ is tight. That is, for any positive real number ϵ, there is a compact set K_ϵ in $Z \times X$ such that for any $n \geq 1$, $f_n(y) \otimes \delta_{x_n}(K_\epsilon) > 1 - \epsilon$. Thus, $\sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i}(K_\epsilon) > 1 - \epsilon$ for any $n \geq 1$ and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Hence, $J(y)$ is compact by Theorem 5.22 of Aliprantis and Border (2006) again.

For any $n \geq 1$ and for any $\alpha_i \in \mathbb{Q}_+$, $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, it is clear that $\sum_{i=1}^{n} \alpha_i f_i(y) \otimes \delta_{x_i}$ is measurable in $y \in Y$. Lemma 2(v) implies that J is also a measurable correspondence from Y to $\mathcal{M}(Z \times X)$. By the argument in the second paragraph of the proof of part (iv), the set

$$\Lambda = \{\lambda \diamond \zeta \colon \zeta \text{ is a Borel measurable selection of } J\}$$

is compact.

Since $\lambda \diamond (f_n \otimes \delta_{x_n}) \in \Lambda$ for each n, there exists some Borel measurable selection ζ of J such that a subsequence of $\{\lambda \diamond (f_n \otimes \delta_{x_n})\}$, say itself, weakly converges to $\lambda \diamond \zeta \in \Lambda$. Let $\zeta_X(y)$ be the marginal probability of $\zeta(y)$ on X for each y. Since x_n converges to x_0, $\zeta_X(y) = \delta_{x_0}$ for λ-almost all $y \in Y$. As a result, there exists a Borel measurable function f_0 such that $\zeta = f_0 \otimes \delta_{x_0}$, where $f_0(y) \in \overline{\text{co}}\,\text{Ls}_n\{f_n(y)\}$ for λ-almost all $y \in Y$. Since F is convex and compact valued, and upper hemicontinuous on X, f_0 is a measurable selection of $F(x_0, \cdot)$. Let $\mu_0 = \lambda \diamond f_0$. Then μ_n weakly converges to μ_0, which implies that Π is upper hemicontinuous.

Next we show the lower hemicontinuity of Π. Suppose that $x_n \to x_0$ and $\mu_0 \in \Pi(x_0)$. Then there exists a Borel measurable selection f_0 of $F(x_0, \cdot)$ such that $\mu_0 = \lambda \diamond f_0$. Since F is sectionally lower hemicontinuous on X and compact-valued, for each $n \geq 1$, there exists a measurable selection f_n for $F(x_n, \cdot)$ such that $f_n(y)$ weakly converges to $f_0(y)$ for each $y \in Y$.[43] Denote $\mu_n = \lambda \diamond f_n$. For any bounded continuous function ψ on $Y \times Z$, $\int_Z \psi(y, z) f_n(dz|y)$ converges to $\int_Z \psi(y, z) f_0(dz|y)$ for any $y \in Y$. Thus, we have

$$\int_{Y\times Z} \psi(y, z)\mu_n\big(d(y, z)\big) = \int_Y \int_Z \psi(y, z) f_n(dz|y)\lambda(dy)$$

[43]See Proposition 4.2 in Sun (1997). Note that the atomless Loeb probability measurable space assumption is not needed for the result of lower hemicontinuity as in Theorem 10 therein.

$$\to \int_Y \int_Z \psi(y,z) f_0(dz|y)\lambda(dy)$$

by the dominated convergence theorem. Therefore, Π is lower hemicontinuous. The proof is thus complete. □

The following lemma is taken from Simon and Zame (1990) (see also Lemma 4 in Reny and Robson (2002)).

- The S and Y are Polish spaces, and D is a closed subset of $S \times Y$, where $D(s)$ is compact for all $s \in S$.
- We have $X = \prod_{1\le i\le n} X_i$, where each X_i is a Polish space.
- For each i, A_i is a nonempty and compact valued continuous correspondence from D to X_i, $A = \prod_{1\le i\le n} A_i$, and $E = \mathrm{Gr}(A)$.
- The P is a bounded, nonempty, convex and compact valued upper hemicontinuous correspondence from E to $\mathbb{R}^n$.

Lemma 10. *Consider the correspondence $\Phi\colon D \to \mathbb{R}^n \times \mathcal{M}(X) \times \triangle(X)$ defined as $(v,\alpha,\mu) \in \Phi(s,y)$ if p is a selection of P such that $p(s,y,\cdot)$ is Borel measurable for any (s,y) and the following relationships hold:*

(i) We have $v = \int_X p(s,y,x)\alpha(dx)$.

(ii) We have that $\alpha \in \bigotimes_{i\in I} \mathcal{M}(A_i(s,y))$ is a Nash equilibrium in the subgame (s,y) with payoff profile $p(s,y,\cdot)$ and action space $A_i(s,y)$ for each player i.

(iii) We have $\mu = p(s,y,\cdot)\circ\alpha$.

Then Φ is nonempty and compact valued, and upper hemicontinuous on D.

A.2 *Proof of Theorem 1*

A.2.1 *Backward induction* As explained in Section 4.3, the backward induction step aims to show that some desirable properties of the equilibrium payoff correspondences can be preserved when one works backward along the game tree.

Given $t \ge 1$, let Q_{t+1} be a bounded, nonempty and compact valued, and upper hemicontinuous correspondence from H_t to $\mathbb{R}^n$. For any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, let

$$P_t(h_{t-1},x_t) = \int_{S_t} Q_{t+1}(h_{t-1},x_t,s_t) f_{t0}(ds_t|h_{t-1}).$$

Since $f_{t0}(\cdot|h_{t-1})$ is atomless and Q_{t+1} is nonempty and compact valued, by Lemma 4,

$$P_t(h_{t-1},x_t) = \int_{S_t} \mathrm{co}\, Q_{t+1}(h_{t-1},x_t,s_t) f_{t0}(ds_t|h_{t-1}),$$

where $\mathrm{co}\, Q_{t+1}(h_{t-1},x_t,s_t)$ is the convex hull of $Q_{t+1}(h_{t-1},x_t,s_t)$. By Lemma 4, P_t is bounded, nonempty, convex and compact valued. To show that P_t is upper hemicontinuous on $\mathrm{Gr}(A_t)$, by Lemma 1(v), it is sufficient to show that P_t is upper hemicontinuous

on $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$, where $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ is a sequence such that $(h_{t-1}^k, x_t^k) \to (h_{t-1}^0, x_t^0)$ as $k \to \infty$. Note that $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ is indeed a compact set. Then the upper hemicontinuity of P_t on $\{(h_{t-1}^k, x_t^k)\}_{k\geq 0}$ follows from Lemma 7.

By Lemma 10, there exists a bounded, measurable, nonempty and compact valued correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ such that Φ_t is upper hemicontinuous, and for all $h_{t-1} \in H_{t-1}$, $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if p_t is a selection of P_t such that $p_t(h_{t-1}, \cdot)$ is Borel measurable and we have the following relationships:

- We have $v = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x)\alpha(dx)$.
- We have that $\alpha \in \bigotimes_{i\in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$.
- We have $\mu = p_t(h_{t-1}, \cdot) \circ \alpha$.

Denote the restriction of Φ_t on the first component $\mathbb{R}^n$ as $\Phi(Q_{t+1})$, which is a correspondence from H_{t-1} to $\mathbb{R}^n$. By Lemma 10, $\Phi(Q_{t+1})$ is bounded, nonempty and compact valued, and upper hemicontinuous.

A.2.2 *Forward induction* If one views Q_t as some payoff correspondence for the players in stage t, then the correspondence Φ_t obtained in the backward induction step collects all the equilibrium strategies, the corresponding payoff vectors, and the induced probabilities in stage $t-1$. The issue here is that one needs to construct jointly measurable payoff functions (as selections of Q_t) and strategy profiles in stage t that induce the equilibrium payoffs in Φ_t. This is done in the forward induction step. Specifically, we prove the following proposition.

Proposition 3. *For any $t \geq 1$ and any Borel measurable selection q_t of $\Phi(Q_{t+1})$, there exists a Borel measurable selection q_{t+1} of Q_{t+1} and a Borel measurable mapping $f_t\colon H_{t-1} \to \bigotimes_{i\in I} \mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in H_{t-1}$, we have the following relationships:*

(i) We have $f_t(h_{t-1}) \in \bigotimes_{i\in I} \mathcal{M}(A_{ti}(h_{t-1}))$.

(ii) We have $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}) f_t(dx_t|h_{t-1})$.

(iii) We have that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action spaces $\{A_{ti}(h_{t-1})\}_{i\in I}$ and payoff functions

$$\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1}).$$

Proof. We divide the proof into three steps. In Step 1, we show that there exist Borel measurable mappings $f_t\colon H_{t-1} \to \bigotimes_{i\in I} \mathcal{M}(X_{ti})$ and $\mu_t\colon H_{t-1} \to \triangle(X_t)$ such that (q_t, f_t, μ_t) is a selection of Φ_t. In Step 2, we obtain a Borel measurable selection g_t of P_t such that for all $h_{t-1} \in H_{t-1}$, we have the following relationships.

- We have $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} g_t(h_{t-1}, x) f_t(dx|h_{t-1})$.

- We have that $f_t(h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with payoff $g_t(h_{t-1}, \cdot)$ and action space $A_t(h_{t-1})$.[44]

In Step 3, we show that there exists a Borel measurable selection q_{t+1} of Q_{t+1} such that for all $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$g_t(h_{t-1}, x_t) = \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}).$$

Combining Steps 1–3 completes the proof.

Step 1. Let $\Psi_t\colon \operatorname{Gr}(\Phi(Q_{t+1})) \to \mathcal{M}(X_t) \times \triangle(X_t)$ be

$$\Psi_t(h_{t-1}, v) = \big\{(\alpha, \mu)\colon (v, \alpha, \mu) \in \Phi_t(h_{t-1})\big\}.$$

For any $\{(h_{t-1}^k, v^k)\}_{1\le k\le\infty} \subseteq \operatorname{Gr}(\Phi(Q_{t+1}))$ such that (h_{t-1}^k, v^k) converges to $(h_{t-1}^\infty, v^\infty)$, pick (α^k, μ^k) such that $(v^k, \alpha^k, \mu^k) \in \Phi_t(h_{t-1}^k)$ for $1 \le k < \infty$. Since Φ_t is upper hemicontinuous and compact-valued, there exists a subsequence of (v^k, α^k, μ^k), say itself, such that (v^k, α^k, μ^k) converges to some $(v^\infty, \alpha^\infty, \mu^\infty) \in \Phi_t(h_{t-1}^\infty)$ due to Lemma 1(v). Thus, $(\alpha^\infty, \mu^\infty) \in \Psi_t(h_{t-1}^\infty, v^\infty)$, which implies that Ψ_t is also upper hemicontinuous and compact-valued. By Lemma 2(iii), Ψ_t has a Borel measurable selection ψ_t. Given a Borel measurable selection q_t of $\Phi(Q_{t+1})$, let $\phi_t(h_{t-1}) = (q_t(h_{t-1}), \psi_t(h_{t-1}, q_t(h_{t-1})))$. Then ϕ_t is a Borel measurable selection of Φ_t. Let f_t and μ_t be the second and third dimensions of ϕ_t, respectively. By the construction of Φ_t, for all $h_{t-1} \in H_{t-1}$, the following relationships hold:

- We have that $q_t(h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x) f_t(dx|h_{t-1})$ such that $p_t(h_{t-1}, \cdot)$ is a Borel measurable selection of $P_t(h_{t-1}, \cdot)$.
- We have that $f_t(h_{t-1}) \in \bigotimes_{i\in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $p_t(h_{t-1}, \cdot)$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$.
- We have $\mu_t(\cdot|h_{t-1}) = p_t(h_{t-1}, \cdot) \circ f_t(\cdot|h_{t-1})$.

Step 2. Since P_t is upper hemicontinuous on $\{(h_{t-1}, x_t)\colon h_{t-1} \in H_{t-1}, x_t \in A_t(h_{t-1})\}$, due to Lemma 6, there exists a Borel measurable mapping g such that (a) $g(h_{t-1}, x_t) \in P_t(h_{t-1}, x_t)$ for any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$, and (b) $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t.

In a subgame $h_{t-1} \in H_{t-1}$, let

$$\begin{aligned} B_{ti}(h_{t-1}) = \Big\{ y_i \in A_{ti}(h_{t-1})\colon & \\ & \int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, y_i, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ & > \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) \Big\}. \end{aligned}$$

[44]One cannot simply use p_t in the previous subsection instead of g_t here. Note that p_t may not be jointly Borel measurable in (h_{t-1}, x) even though $p_t(h_{t-1}, \cdot)$ is Borel measurable for each fixed h_{t-1}.

Since $g(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t,

$$\int_{A_t(h_{t-1})} g(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) = \int_{A_t(h_{t-1})} p_t(h_{t-1}, x_t) f_t(dx_t|h_{t-1}).$$

Thus, B_{ti} is a measurable correspondence from H_{t-1} to $A_{ti}(h_{t-1})$. Then B_{ti} has a Borel graph. As $f_t(h_{t-1})$ is a Nash equilibrium in the subgame $h_{t-1} \in H_{t-1}$ with payoff $p_t(h_{t-1}, \cdot)$, $f_{ti}(B_{ti}(h_{t-1})|h_{t-1}) = 0$.

Denote $\beta_i(h_{t-1}, x_t) = \min P_{ti}(h_{t-1}, x_t)$, where $P_{ti}(h_{t-1}, x_t)$ is the projection of $P_t(h_{t-1}, x_t)$ on the ith dimension. Then the correspondence P_{ti} is measurable and compact-valued, and β_i is Borel measurable. Let $\Lambda_i(h_{t-1}, x_t) = \{\beta_i(h_{t-1}, x_t)\} \times [0, \gamma]^{n-1}$, where $\gamma > 0$ is the upper bound of P_t. Denote $\Lambda_i'(h_{t-1}, x_t) = \Lambda_i(h_{t-1}, x_t) \cap P_t(h_{t-1}, x_t)$. Then Λ_i' is a measurable and compact-valued correspondence, and hence has a Borel measurable selection β_i'. Note that β_i' is a Borel measurable selection of P_t. Let

$$g_t(h_{t-1}, x_t) = \begin{cases} \beta_i'(h_{t-1}, x_t) & \text{if } h_{t-1} \in H_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i, \\ g(h_{t-1}, x_t) & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} &\big\{(h_{t-1}, x_t) \in \mathrm{Gr}(A_t) \colon h_{t-1} \in H_{t-1}, x_{ti} \in B_{ti}(h_{t-1}) \text{ and } x_{tj} \notin B_{tj}(h_{t-1}), \forall j \neq i; \big\} \\ &\quad = \mathrm{Gr}(A_t) \cap \bigcup_{i \in I} \Bigg(\Bigg(\mathrm{Gr}(B_{ti}) \times \prod_{j \neq i} X_{tj} \Bigg) \Bigg\backslash \Bigg(\bigcup_{j \neq i} \Bigg(\mathrm{Gr}(B_{tj}) \times \prod_{k \neq j} X_{tk} \Bigg) \Bigg) \Bigg), \end{aligned}$$

which is a Borel set. As a result, g_t is a Borel measurable selection of P_t. Moreover, $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for all $h_{t-1} \in H_{t-1}$ and $f_t(\cdot|h_{t-1})$-almost all x_t.

Fix a subgame $h_{t-1} \in H_{t-1}$. We verify that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}. Suppose that player i deviates to some action $\tilde{x}_{ti}$.

If $\tilde{x}_{ti} \in B_{ti}(h_{t-1})$, then player i's expected payoff is

$$\begin{aligned} &\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\quad = \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\quad = \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\quad \leq \int_{\prod_{j \neq i} B_{tj}^c(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\quad = \int_{A_{t(-i)}(h_{t-1})} p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\ &\quad \leq \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) \end{aligned}$$

$$= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}).$$

The first and the third equalities hold since $f_{tj}(B_{tj}(h_{t-1})|h_{t-1}) = 0$ for each j, and hence $f_{t(-i)}(\prod_{j\neq i} B^c_{tj}(h_{t-1})|h_{t-1}) = f_{t(-i)}(A_{t(-i)}(h_{t-1})|h_{t-1})$. The second equality and the first inequality are due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \beta_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = \min P_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) \leq p_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j\neq i} B^c_{tj}(h_{t-1})$. The second inequality holds since $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $p_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}. The fourth equality follows from the fact that $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t.

If $\tilde{x}_{ti} \notin B_{ti}(h_{t-1})$, then player i's expected payoff is

$$\begin{aligned}
&\int_{A_{t(-i)}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{\prod_{j\neq i} B^c_{tj}(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&= \int_{A_{t(-i)}(h_{t-1})} g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) f_{t(-i)}(dx_{t(-i)}|h_{t-1}) \\
&\leq \int_{A_t(h_{t-1})} p_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}) \\
&= \int_{A_t(h_{t-1})} g_{ti}(h_{t-1}, x_t) f_t(dx_t|h_{t-1}).
\end{aligned}$$

The first and the third equalities hold since

$$f_{t(-i)}\Big(\prod_{j\neq i} B^c_{tj}(h_{t-1})|h_{t-1}\Big) = f_{t(-i)}\big(A_{t(-i)}(h_{t-1})|h_{t-1}\big).$$

The second equality is due to the fact that $g_{ti}(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)}) = g_i(h_{t-1}, \tilde{x}_{ti}, x_{t(-i)})$ for $x_{t(-i)} \in \prod_{j\neq i} B^c_{tj}(h_{t-1})$. The first inequality follows from the definition of B_{ti}, and the fourth equality holds since $g_t(h_{t-1}, x_t) = p_t(h_{t-1}, x_t)$ for $f_t(\cdot|h_{t-1})$-almost all x_t.

Thus, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy for any $i \in I$, which implies that $f_t(\cdot|h_{t-1})$ is a Nash equilibrium given the payoff $g_t(h_{t-1}, \cdot)$ in the subgame h_{t-1}.

Step 3. For any $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}).$$

By Lemma 5, there exists a Borel measurable mapping q from $\mathrm{Gr}(P_t) \times S_t$ to $\mathbb{R}^n$ such that we have the following relationships:

- We have $q(h_{t-1}, x_t, e, s_t) \in Q_{t+1}(h_{t-1}, x_t, s_t)$ for any $(h_{t-1}, x_t, e, s_t) \in \mathrm{Gr}(P_t) \times S_t$.

- We have $e = \int_{S_t} q(h_{t-1}, x_t, e, s_t) f_{t0}(ds_t|h_{t-1})$ for any $(h_{t-1}, x_t, e) \in \mathrm{Gr}(P_t)$, where $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$.

Let

$$q_{t+1}(h_{t-1}, x_t, s_t) = q\big(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t\big)$$

for any $(h_{t-1}, x_t, s_t) \in H_t$. Then q_{t+1} is a Borel measurable selection of Q_{t+1}.

For $(h_{t-1}, x_t) \in \mathrm{Gr}(A_t)$,

$$\begin{aligned} g_t(h_{t-1}, x_t) &= \int_{S_t} q\big(h_{t-1}, x_t, g_t(h_{t-1}, x_t), s_t\big) f_{t0}(ds_t|h_{t-1}) \\ &= \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}). \end{aligned}$$

Therefore, we have a Borel measurable selection q_{t+1} of Q_{t+1} and a Borel measurable mapping $f_t\colon H_{t-1} \to \bigotimes_{i\in I} \mathcal{M}(X_{ti})$ such that for all $h_{t-1} \in H_{t-1}$, properties (i)–(iii) are satisfied. The proof is complete. □

If a dynamic game has only T stages for some positive integer $T \geq 1$, then let $Q_{T+1}(h_T) = \{u(h_T)\}$ for any $h_T \in H_T$ and $Q_t = \Phi(Q_{t+1})$ for $1 \leq t \leq T-1$. We can start with the backward induction from the last period and stop at the initial period, and then run the forward induction from the initial period to the last period. We obtain the following corollary.

Corollary 1. *If a finite-horizon continuous dynamic game with almost perfect information satisfies the condition of atomless transitions, then it has a subgame-perfect equilibrium.*[45]

A.2.3 *Infinite-horizon case* Pick a sequence $\xi = (\xi_1, \xi_2, \ldots)$ such that (a) ξ_m is a transition probability from H_{m-1} to $\mathcal{M}(X_m)$ for any $m \geq 1$, and (b) $\xi_m(A_m(h_{m-1})|h_{m-1}) = 1$ for any $m \geq 1$ and $h_{m-1} \in H_{m-1}$. Denote the set of all such ξ by Υ. Intuitively, ξ can be viewed as a correlated strategy profile with each ξ_t being the correlated strategy in stage t, and Υ is the set of all such correlated strategies.

Fix any $t \geq 1$ and define a correspondence Δ_t^t as follows: in the subgame h_{t-1},

$$\Delta_t^t(h_{t-1}) = \mathcal{M}\big(A_t(h_{t-1})\big) \otimes f_{t0}(h_{t-1}).$$

Then $\Delta_t^t(h_{t-1})$ is the set of probability measures on the space of action profiles of stage t, which is induced by all the possible correlated strategies among the active players and Nature's move in the subgame h_{t-1}. Inductively, we define the set of possible paths for correlated strategies in any subgame between stages t and m_1 for $t < m_1 \leq \infty$.

For any integer $m_1 > t$, suppose that the correspondence $\Delta_t^{m_1-1}$ has been defined. The correspondence $\Delta_t^{m_1}\colon H_{t-1} \to \mathcal{M}(\prod_{t\leq m\leq m_1}(X_m \times S_m))$ is defined as

$$\Delta_t^{m_1}(h_{t-1}) = \big\{g(h_{t-1}) \diamond \big(\xi_{m_1}(h_{t-1}, \cdot) \otimes f_{m_1 0}(h_{t-1}, \cdot)\big)\colon$$

[45]The condition of atomless transition is not needed at the last stage.

$$g \text{ is a Borel measurable selection of } \Delta_t^{m_1-1},$$
$$\xi_{m_1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_1})\big\},$$

where the correspondence $\mathcal{M}(A_{m_1})$ takes value $\mathcal{M}(A_{m_1}(h_{m_1-1}))$ at subgame h_{m_1-1}. For any $m_1 \geq t$, let $\varrho^{m_1}_{(h_{t-1},\xi)} \in \Delta_t^{m_1}$ be the probability measure on $\prod_{t\leq m\leq m_1}(X_m \times S_m)$ induced by Nature's moves $\{f_{m0}\}_{t\leq m\leq m_1}$ and the correlated strategies $\{\xi_m\}_{t\leq m\leq m_1}$. Then $\varrho^{m_1}_{(h_{t-1},\xi)}$ is a possible path induced by ξ in the subgame h_{t-1} before stage m_1, and $\Delta_t^{m_1}(h_{t-1})$ is the set of all such possible paths $\varrho^{m_1}_{(h_{t-1},\xi)}$ in the subgame h_{t-1}. Note that $\varrho^{m_1}_{(h_{t-1},\xi)}$ can be regarded as a probability measure on $H_{m_1}(h_{t-1}) = \{(x_t, s_t, \ldots, x_{m_1}, s_{m_1}) : (h_{t-1}, x_t, s_t, \ldots, x_{m_1}, s_{m_1}) \in H_{m_1}\}$. Similarly, let $\varrho_{(h_{t-1},\xi)}$ be the probability measure on $\prod_{m\geq t}(X_m \times S_m)$ induced by Nature's moves $\{f_{m0}\}_{m\geq t}$ and the correlated strategies $\{\xi_m\}_{m\geq t}$ after the subgame h_{t-1}. The correspondence

$$\Delta_t : H_{t-1} \to \mathcal{M}\Big(\prod_{m\geq t}(X_m \times S_m)\Big)$$

collects all the possible paths $\varrho_{(h_{t-1},\xi)}$.

We show that the correspondence $\Delta_t(h_{t-1})$, which contains all the possible paths induced by correlated strategies in the subgame h_{t-1}, is nonempty and compact valued, and continuous. The claim is proved by showing that $\Delta_t^{m_1}$ has such properties and that Δ_t can be approximated by $\Delta_t^{m_1}$ when m_1 is sufficiently large.

Lemma 11. *(i) For any $t \geq 1$, the correspondence $\Delta_t^{m_1}$ is nonempty and compact valued, and continuous for any $m_1 \geq t$.*

(ii) For any $t \geq 1$, the correspondence Δ_t is nonempty and compact valued, and continuous.

Proof. (i) Consider the case $m_1 = t \geq 1$, where

$$\Delta_t^{m_1}(h_{t-1}) = \mathcal{M}\big(A_t(h_{t-1})\big) \otimes f_{t0}(h_{t-1}).$$

Since A_t is nonempty and compact valued, and both A_t and f_{t0} are continuous, $\Delta_t^{m_1}$ is nonempty compact valued, and continuous.

Suppose that $\Delta_t^{m_2}$ is nonempty and compact valued, and continuous for some $m_2 \geq t \geq 1$. Note that

$$\Delta_t^{m_2+1}(h_{t-1}) = \big\{g(h_{t-1}) \diamond \big(\xi_{m_2+1}(h_{t-1}, \cdot) \otimes f_{(m_2+1)0}(h_{t-1}, \cdot)\big) :$$
$$g \text{ is a Borel measurable selection of } \Delta_t^{m_2},$$
$$\xi_{m_2+1} \text{ is a Borel measurable selection of } \mathcal{M}(A_{m_2+1})\big\}.$$

By Lemma 9(iii), $\Delta_t^{m_2+1}$ is nonempty and compact valued, and continuous.

(ii) It is obvious that Δ_t is nonempty-valued. We first prove that it is upper hemicontinuous and compact-valued.

Given sequences $\{h_{t-1}^k\}$ and $\tau^k \subseteq \Delta_t(h_{t-1}^k)$, there exists a sequence of $\{\xi^k\}_{k\geq 1}$ such that $\xi^k = (\xi_1^k, \xi_2^k, \ldots) \in \Upsilon$ and $\tau^k = \varrho_{(h_{t-1}^k, \xi^k)}$ for each $k \geq 1$. Suppose that $h_{t-1}^k \to h_{t-1}^\infty$. By (i), Δ_t^t is compact-valued and upper hemicontinuous. Then there exists a measurable mapping g_t such that (a) $g^t = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, \xi_{t+1}^1, \ldots) \in \Upsilon$ and (b) a subsequence of $\{\varrho^t_{(h_{t-1}^k, \xi^k)}\}$, say $\{\varrho^t_{(h_{t-1}^{k_{1l}}, \xi^{k_{1l}})}\}_{l\geq 1}$, that weakly converges to $\varrho^t_{(h_{t-1}^\infty, g^t)}$. Note that $\{\xi_{t+1}^k\}$ is a Borel measurable selection of $\mathcal{M}(A_{t+1})$. By Lemma 9(iii), there is a Borel measurable selection g_{t+1} of $\mathcal{M}(A_{t+1})$ such that there is a subsequence of $\{\varrho^{t+1}_{(h_{t-1}^{k_{1l}}, \xi^{k_{1l}})}\}_{l\geq 1}$, say $\{\varrho^{t+1}_{(h_{t-1}^{k_{2l}}, \xi^{k_{2l}})}\}_{l\geq 1}$, that weakly converges to $\varrho^{t+1}_{(h_{t-1}^\infty, g^{t+1})}$, where $g^{t+1} = (\xi_1^1, \ldots, \xi_{t-1}^1, g_t, g_{t+1}, \xi_{t+2}^1, \ldots) \in \Upsilon$. Repeating this procedure, one can construct a Borel measurable mapping g such that $\varrho_{(h_{t-1}^{k_{11}}, \xi^{k_{11}})}, \varrho_{(h_{t-1}^{k_{22}}, \xi^{k_{22}})}, \varrho_{(h_{t-1}^{k_{33}}, \xi^{k_{33}})}, \ldots$ weakly converges to $\varrho_{(h_{t-1}^\infty, g)}$. That is, $\varrho_{(h_{t-1}^\infty, g)}$ is a convergent point of $\{\varrho_{(h_{t-1}^k, \xi^k)}\}$. By Lemma 1(v), Δ_t is compact-valued and upper hemicontinuous.

Next we consider the lower hemicontinuity of Δ_t. Suppose that $\tau^0 \in \Delta_t(h_{t-1}^0)$. Then there exists some $\xi \in \Upsilon$ such that $\tau^0 = \varrho_{(h_{t-1}^0, \xi)}$. Denote $\tilde{\tau}^m = \varrho^m_{(h_{t-1}^0, \xi)} \in \Delta_t^m(h_{t-1}^0)$ for $m \geq t$. As Δ_t^m is continuous, for each m, there exists some $\xi^m \in \Upsilon$ such that $d(\varrho^m_{(h_{t-1}^{k_m}, \xi^m)}, \tilde{\tau}^m) \leq \frac{1}{m}$ for k_m sufficiently large, where d is the Prokhorov metric. Let $\tau^m = \varrho_{(h_{t-1}^{k_m}, \xi^m)}$. Then τ^m weakly converges to τ^0, which implies that Δ_t is lower hemicontinuous. □

Below, we define a correspondence Q_t^τ from H_{t-1} to $\mathbb{R}^n_{++}$ inductively for any stages $t, \tau \geq 1$. When $\tau < t$, $Q_t^\tau(h_{t-1})$ is the set of all the payoffs based on correlated strategies in the subgame h_{t-1}, which does not depend on τ. As a result, for any stage τ, $Q_{\tau+1}^\tau$ can be defined. Then for $\tau \geq t$, Q_t^τ is the correspondence obtained by repeating the backward induction from the correspondence $Q_{\tau+1}^\tau$. Specifically,

$$Q_t^\tau(h_{t-1}) = \begin{cases} \left\{ \int_{\prod_{m\geq t}(X_m \times S_m)} u(h_{t-1}, x, s)\varrho_{(h_{t-1},\xi)}\big(d(x,s)\big) \colon \varrho_{(h_{t-1},\xi)} \in \Delta_t(h_{t-1}) \right\}, & \tau < t, \\ \Phi\big(Q_{t+1}^\tau\big)(h_{t-1}), & \tau \geq t. \end{cases}$$

The lemma below presents several desirable properties of Q_t^τ.

Lemma 12. *For any $t, \tau \geq 1$, Q_t^τ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.*

Proof. For $t > \tau$, Q_t^τ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous because of the corresponding properties of u and Δ_t.

For $t \leq \tau$, we can start with $Q_{\tau+1}^\tau$. Repeating the backward induction in Appendix A.2.1, Q_t^τ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous. □

Denote $Q_t^\infty = \bigcap_{\tau\geq 1} Q_t^\tau$. Recall that Q_t^τ is the payoff correspondence of correlated strategies when $\tau < t$ and the correspondence obtained by repeating the backward induction from $Q_{\tau+1}^\tau$ when $\tau \geq t$. Given some t, when τ is sufficiently large, it is expected that $Q_{\tau+1}^\tau$ should be close to the payoff correspondence of all the mixed strategies, as the game is continuous at infinity. Given the correspondence $Q_{\tau+1}^\tau$, players play some equilibrium strategies in each backward induction step. As a result, it is expected that Q_t^τ would be close to the actual equilibrium payoff correspondence E_t for sufficiently large τ. The following three lemmas show that $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1}) = E_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.[46]

Lemma 13. *(i) The correspondence Q_t^∞ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous.*

(ii) For any $t \geq 1$, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

Proof. (i) It is obvious that Q_t^∞ is bounded. By the definition of Q_t^τ, for all $h_{t-1} \in H_{t-1}$, $Q_t^{\tau_1}(h_{t-1}) \subseteq Q_t^{\tau_2}(h_{t-1})$ for $\tau_1 \geq \tau_2$. Since Q_t^τ is nonempty and compact valued, $Q_t^\infty = \bigcap_{\tau\geq 1} Q_t^\tau$ is nonempty and compact valued. By Lemma 2(ii), $\bigcap_{\tau\geq 1} Q_t^\tau$ is measurable, which implies that Q_t^∞ is measurable.

Since Q_t^τ is upper hemicontinuous for any τ, by Lemma 2(vii), it has a closed graph for each τ, which implies that Q_t^∞ has a closed graph. Referring to Lemma 2(vii) again, Q_t^∞ is upper hemicontinuous.

(ii) For any $\tau \geq 1$ and $h_{t-1} \in H_{t-1}$, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq \Phi(Q_{t+1}^\tau)(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$, and, hence, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$.

Let $\{1, 2, \ldots \infty\}$ be a countable compact space endowed with the metric $d(k, m) = |\frac{1}{k} - \frac{1}{m}|$ for $1 \leq k, m \leq \infty$. The sequence $\{Q_{t+1}^\tau\}_{1\leq\tau\leq\infty}$ can be regarded as a correspondence Q_{t+1} from $H_t \times \{1, 2, \ldots, \infty\}$ to $\mathbb{R}^n$, which is measurable, nonempty and compact valued, and upper hemicontinuous on $H_t \times \{1, 2, \ldots, \infty\}$. The step of backward induction in Appendix A.2.1 shows that $\Phi(Q_{t+1})$ is measurable, nonempty and compact valued, and upper hemicontinuous on $H_t \times \{1, 2, \ldots, \infty\}$. For $h_{t-1} \in H_{t-1}$ and $a \in Q_t^\infty(h_{t-1})$, by its definition, $a \in Q_t^\tau(h_{t-1}) = \Phi(Q_{t+1}^\tau)(h_{t-1})$ for $\tau \geq t$. Thus, $a \in \Phi(Q_{t+1}^\infty)(h_{t-1})$.

As a result, $Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. □

Though the definition of Q_t^τ involves correlated strategies for $\tau < t$, the following lemma shows that one can work with mixed strategies in terms of equilibrium payoffs. In Lemma 13, it is shown that $Q_t^\infty = \Phi(Q_{t+1}^\infty)$ for any $t \geq 1$. Then one can apply the forward induction recursively to obtain the mixed strategies $\{f_{ki}\}_{i\in I}$ and a selection c_k of Q_k^∞ for each $k \geq 1$ such that f_{ki} is an equilibrium and c_k is the corresponding equilibrium payoff given the payoff function c_{k+1}. Lemma 14 shows that $\{f_{ki}\}_{k\geq 1, i\in I}$ is indeed an equilibrium and c_k is the equilibrium payoff of the game in stage k. The key here is that since $Q_k^\infty \subseteq Q_k^\tau = \Phi^{\tau-k+1}(Q_{\tau+1}^\tau)$ for $k \leq \tau$, one can obtain c_k via the strategies

[46]The proofs of Lemmas 13 and 15 follow the standard argument with various modifications; see, for example, Harris (1990), Harris et al. (1995), and Mariotti (2000).

$\{f_{ki}\}_{k\le k'\le\tau, i\in I}$ between stages k and τ, and the correlated strategies $\{\xi_{k'}\}_{k'>\tau}$ after stage τ. Because of the assumption of the continuity at infinity, the latter payoff converges to c_k as $\tau\to\infty$. To check the equilibrium property, one can rely on the same asymptotic argument to check the payoff for any deviation. Then we show that c_k is an equilibrium payoff without using the correlated strategies.

Lemma 14. *If c_t is a measurable selection of $\Phi(Q_{t+1}^{\infty})$, then $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for any $h_{t-1}\in H_{t-1}$.*

Proof. Without loss of generality, we prove only the case $t=1$.

Suppose that c_1 is a measurable selection of $\Phi(Q_2^{\infty})$. Apply Proposition 3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i\in I}$ for $k\ge 1$. That is, for any $k\ge 1$, there exists a Borel measurable selection c_k of Q_k^{∞} such that for all $h_{k-1}\in H_{k-1}$, we have the following relationships:

- We have that $f_k(h_{k-1})$ is a Nash equilibrium in the subgame h_{k-1}, where the action space is $A_{ki}(h_{k-1})$ for player $i\in I$, and the payoff function is given by
$$\int_{S_k} c_{k+1}(h_{k-1},\cdot,s_k)f_{k0}(ds_k|h_{k-1}).$$
- We have
$$c_k(h_{k-1})=\int_{A_k(h_{k-1})}\int_{S_k} c_{k+1}(h_{k-1},x_k,s_k)f_{k0}(ds_k|h_{k-1})f_k(dx_k|h_{k-1}).$$

We need to show that $c_1(h_0)$ is a subgame-perfect equilibrium payoff vector for all $h_0\in H_0$.

Step 1. We show that for any $k\ge 1$ and all $h_{k-1}\in H_{k-1}$,
$$c_k(h_{k-1})=\int_{\prod_{m\ge k}(X_m\times S_m)} u(h_{k-1},x,s)\varrho_{(h_{k-1},f)}\big(d(x,s)\big).$$

Fix a positive integer $M>k$. By Lemma 13, $c_k(h_{k-1})\in Q_k^{\infty(h_{k-1})=\bigcap_{\tau\ge 1}Q_k^{\tau}}\times(h_{k-1})$ for all $h_{k-1}\in H_{k-1}$. Since $Q_k^{\tau}=\Phi^{\tau-k+1}(Q_{\tau+1}^{\tau})$ for $k\le\tau$, $c_k(h_{k-1})\in\bigcap_{\tau\ge k}\Phi^{\tau-k+1}\times(Q_{\tau+1}^{\tau})(h_{k-1})\subseteq\Phi^{M-k+1}(Q_{M+1}^{M})(h_{k-1})$ for all $h_{k-1}\in H_{k-1}$. Thus, there exists a Borel measurable selection w of Q_{M+1}^{M} and some $\xi\in\Upsilon$ such that for all $h_{M-1}\in H_{M-1}$, we have the following relationships:

(i) We have that $f_M(h_{M-1})$ is a Nash equilibrium in the subgame h_{M-1}, where the action space is $A_{Mi}(h_{M-1})$ for player $i\in I$ and the payoff function is given by
$$\int_{S_M} w(h_{M-1},\cdot,s_M)f_{M0}(ds_M|h_{M-1}).$$

(ii) We have
$$c_M(h_{M-1})=\int_{A_M(h_{M-1})}\int_{S_M} w(h_{M-1},x_M,s_M)f_{M0}(ds_M|h_{M-1})f_M(dx_M|h_{M-1}).$$

(iii) We have $w(h_M) = \int_{\prod_{m\geq M+1}(X_m\times S_m)} u(h_M, x, s)\varrho_{(h_M,\xi)}(d(x,s))$.

Then for $h_{k-1} \in H_{k-1}$,

$$c_k(h_{k-1}) = \int_{\prod_{m\geq k}(X_m\times S_m)} u(h_{k-1}, x, s)\varrho_{(h_{k-1},f^M)}\big(d(x,s)\big),$$

where f_k^M is f_k if $k \leq M$ and is ξ_k if $k \geq M+1$. Since the game is continuous,

$$\int_{\prod_{m\geq k}(X_m\times S_m)} u(h_{k-1}, x, s)\varrho_{(h_{k-1},f^M)}\big(d(x,s)\big)$$

converges to

$$\int_{\prod_{m\geq k}(X_m\times S_m)} u(h_{k-1}, x, s)\varrho_{(h_{k-1},f)}\big(d(x,s)\big)$$

when M goes to infinity. Thus, for all $h_{k-1} \in H_{k-1}$,

$$c_k(h_{t-1}) = \int_{\prod_{m\geq k}(X_m\times S_m)} u(h_{k-1}, x, s)\varrho_{(h_{k-1},f)}\big(d(x,s)\big). \tag{2}$$

Step 2. Below, we show that $\{f_{ki}\}_{i\in I}$ is a subgame-perfect equilibrium.

Fix a player i and a strategy $g_i = \{g_{ki}\}_{k\geq 1}$. For each $k \geq 1$, define a new strategy $\tilde{f}_i^k$ as $\tilde{f}_i^k = (g_{1i}, \ldots, g_{ki}, f_{(k+1)i}, f_{(k+2)i}, \ldots)$; that is, we simply replace the initial k stages of f_i by g_i. Denote $\tilde{f}^k = (\tilde{f}_i^k, f_{k(-i)})$.

Fix $k \geq 1$. For any $h_k = (x^k, s^k)$, we have

$$\begin{aligned}
&\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,f)}\big(d(x,s)\big)\\
&\quad= \int_{A_{k+1}(h_k)}\int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1}) f_{(k+1)0}(ds_{k+1}|h_k) f_{k+1}(dx_{k+1}|h_k)\\
&\quad\geq \int_{A_{k+1}(h_k)}\int_{S_{k+1}} c_{(k+2)i}(h_k, x_{k+1}, s_{k+1})\\
&\qquad f_{(k+1)0}(ds_{k+1}|h_k)(f_{(k+1)(-i)} \otimes g_{(k+1)i})(dx_{k+1}|h_k)\\
&\quad= \int_{A_{k+1}(h_k)}\int_{S_{k+1}}\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2})\\
&\qquad f_{(k+2)0}(ds_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(dx_{k+2}|h_k, x_{k+1}, s_{k+1})\\
&\qquad f_{(k+1)0}(ds_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(dx_{k+1}|h_k)\\
&\quad\geq \int_{A_{k+1}(h_k)}\int_{S_{k+1}}\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})}\int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2})\\
&\qquad f_{(k+2)0}(ds_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(dx_{k+2}|h_k, x_{k+1}, s_{k+1})\\
&\qquad f_{(k+1)0}(ds_{k+1}|h_k) f_{(k+1)(-i)} \otimes g_{(k+1)i}(dx_{k+1}|h_k)
\end{aligned}$$

$$= \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,\tilde{f}^{k+2})}\big(d(x,s)\big).$$

The first and the last equalities follow from (2) in the end of Step 1. The second equality is due to (ii) in Step 1. The first inequality is based on (i) in Step 1. The second inequality holds since by the choice of h_k and (i) in Step 1, for $f_{(k+1)0}(h_k)$-almost all $s_{k+1} \in S_{k+1}$ and all $x_{k+1} \in X_{k+1}$, we have

$$\begin{aligned}
&\int_{A_{k+2}(h_k,x_{k+1},s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(ds_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes f_{(k+2)i}(dx_{k+2}|h_k, x_{k+1}, s_{k+1}) \\
&\geq \int_{A_{k+2}(h_k,x_{k+1},s_{k+1})} \int_{S_{k+2}} c_{(k+3)i}(h_k, x_{k+1}, s_{k+1}, x_{k+2}, s_{k+2}) \\
&\quad f_{(k+2)0}(ds_{k+2}|h_k, x_{k+1}, s_{k+1}) f_{(k+2)(-i)} \otimes g_{(k+2)i}(dx_{k+2}|h_k, x_{k+1}, s_{k+1}).
\end{aligned}$$

Repeating the above argument, one can show that

$$\begin{aligned}
&\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,f)}\big(d(x,s)\big) \\
&\quad \geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}\big(d(x,s)\big)
\end{aligned}$$

for any $\tilde{M} > k$. Since

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,\tilde{f}^{\tilde{M}+1})}\big(d(x,s)\big)$$

converges to

$$\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,(g_i,f_{-i}))}\big(d(x,s)\big)$$

as $\tilde{M}$ goes to infinity, we have

$$\begin{aligned}
&\int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,f)}\big(d(x,s)\big) \\
&\quad \geq \int_{\prod_{m\geq k+1}(X_m\times S_m)} u(h_k, x, s)\varrho_{(h_k,(g_i,f_{-i}))}\big(d(x,s)\big).
\end{aligned}$$

Therefore, $\{f_{ki}\}_{i\in I}$ is a subgame-perfect equilibrium. □

By Lemmas 10 and 13, the correspondence $\Phi(Q_{t+1}^{\infty})$ is measurable, nonempty and compact valued. By Lemma 2(iii), it has a measurable selection. Then the equilibrium existence result in Theorem 1 follows from the above lemma.

For $t \geq 1$ and $h_{t-1} \in H_{t-1}$, recall that $E_t(h_{t-1})$ is the set of payoff vectors of subgame-perfect equilibria in the subgame h_{t-1}. The following lemma shows that $E_t(h_{t-1})$ is the same as $Q_t^{\infty}(h_{t-1})$.

LEMMA 15. *For any $t \geq 1$, $E_t(h_{t-1}) = Q_t^\infty(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.*

PROOF. Step 1. We first prove the following claim: for any t and τ, if $E_{t+1}(h_t) \subseteq Q_{t+1}^\tau(h_t)$ for all $h_t \in H_t$, then $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. We only need to consider the case that $t \leq \tau$.

By the construction of $\Phi(Q_{t+1}^\tau)$ in Appendix A.2.1, for any c_t and $h_{t-1} = (x^{t-1}, s^{t-1}) \in H_{t-1}$, if the following relationships hold, then $c_t \in \Phi(Q_{t+1}^\tau)(h_{t-1})$.

- We have $c_t = \int_{A_t(h_{t-1})} \int_{S_t} q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1})\alpha(dx_t)$, where $q_{t+1}(h_{t-1}, \cdot)$ is measurable and $q_{t+1}(h_{t-1}, x_t, s_t) \in Q_{t+1}^\tau(h_{t-1}, x_t, s_t)$ for all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$.
- We have that $\alpha \in \bigotimes_{i\in I} \mathcal{M}(A_{ti}(h_{t-1}))$ is a Nash equilibrium in the subgame h_{t-1} with payoff $\int_{S_t} q_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1})$ and action space $\prod_{i\in I} A_{ti}(h_{t-1})$,

Fix a subgame $h_{t-1} = (x^{t-1}, s^{t-1})$. Pick a point $c_t \in E_t(h_{t-1})$. There exists a strategy profile f such that f is a subgame-perfect equilibrium in the subgame h_{t-1} and the payoff is c_t. Let $c_{t+1}(h_{t-1}, x_t, s_t)$ be the payoff vector induced by $\{f_{ti}\}_{i\in I}$ in the subgame $(h_{t-1}, x_t, s_t) \in \mathrm{Gr}(A_t) \times S_t$. Then we have the following relationships:

- We have that $c_t = \int_{A_t(h_{t-1})} \int_{S_t} c_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}) f_t(dx_t|h_{t-1})$.
- We have that $\mathrm{n}f_t(\cdot|h_{t-1})$ is a Nash equilibrium in the subgame h_{t-1} with action space $A_t(h_{t-1})$ and payoff $\int_{S_t} c_{t+1}(h_{t-1}, \cdot, s_t) f_{t0}(ds_t|h_{t-1})$.

Since f is a subgame-perfect equilibrium in the subgame h_{t-1}, $c_{t+1}(h_{t-1}, x_t, s_t) \in E_{t+1}(h_{t-1}, x_t, s_t) \subseteq Q_{t+1}^\tau(h_{t-1}, x_t, s_t)$ for all $s_t \in S_t$ and $x_t \in A_t(h_{t-1})$, which implies that $c_t \in \Phi(Q_{t+1}^\tau)(h_{t-1}) = Q_t^\tau(h_{t-1})$.

Therefore, $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

Step 2. For any $t > \tau$, $E_t \subseteq Q_t^\tau$. If $t \leq \tau$, we can start with $E_{\tau+1} \subseteq Q_{\tau+1}^\tau$ and repeat the argument in Step 1. Then we can show that $E_t(h_{t-1}) \subseteq Q_t^\tau(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. Thus, $E_t(h_{t-1}) \subseteq Q_t^\infty(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

Suppose that c_t is a measurable selection from $\Phi(Q_{t+1}^\infty)$. Apply Proposition 3 recursively to obtain Borel measurable mappings $\{f_{ki}\}_{i\in I}$ for $k \geq t$. By Lemma 14, $c_t(h_{t-1})$ is a subgame-perfect equilibrium payoff vector for all $h_{t-1} \in H_{t-1}$. Consequently, $\Phi(Q_{t+1}^\infty)(h_{t-1}) \subseteq E_t(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$.

By Lemma 13, $E_t(h_{t-1}) = Q_t^\infty(h_{t-1}) = \Phi(Q_{t+1}^\infty)(h_{t-1})$ for all $h_{t-1} \in H_{t-1}$. □

Therefore, we complete the proof of Theorem 1.

A.3 *Proof of Theorem 2*

Step 1: Backward induction. For any $t \geq 1$, suppose that the correspondence Q_{t+1} from H_t to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous on X^t.

If player j is the active player in stage t, then we assume that $S_t = \{\acute{s}_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s}_t)$, which is nonempty and compact valued, and upper hemicontinuous. Note that P_t may not be convex-valued. Then define the correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ as $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if we have the following relationships:

- We have $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$.
- We have that $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$.
- We have $\mu = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj}) \circ \alpha$.

If P_t is nonempty, convex and compact valued, and upper hemicontinuous, then we can use Lemma 10, the main result of Simon and Zame (1990), to prove the nonemptiness, compactness, and upper hemicontinuity of Φ_t. In Simon and Zame (1990), the only step they need the convexity of P_t for the proof of their main theorem is Lemma 2 therein. However, the one-player pure-strategy version of their Lemma 2, stated in the following paragraph, directly follows from the upper hemicontinuity of P_t without requiring the convexity.

> Let Z be a compact metric space, and $\{z_n\}_{n\geq 0} \subseteq Z$. Let $P\colon Z \to \mathbb{R}_+$ be a bounded, upper hemicontinuous correspondence with nonempty and compact values. For each $n \geq 1$, let q_n be a Borel measurable selection of P such that $q_n(z_n) = d_n$. If z_n converges to z_0 and d_n converges to some d_0, then $d_0 \in P(z_0)$.

Repeating the argument in the proof of the main theorem of Simon and Zame (1990), one can show that Φ_t is nonempty and compact valued, and upper hemicontinuous.

Next we consider the case that Nature moves in stage t; that is, there is no active player in I moving in this stage and $A_t(h_{t-1})$ is a singleton set. Suppose that the correspondence Q_{t+1} from H_t to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous. Let

$$P_t(h_{t-1}, x_t) = \int_{S_t} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}),$$

where $A_t(h_{t-1}) = \{x_t\}$. Since $f_{t0}(\cdot|h_{t-1})$ is atomless, as in Appendix A.2.1, P_t is nonempty, convex and compact valued, and upper hemicontinuous. The rest of the step remains the same as in Appendix A.2.1.

In summary, Φ_t is nonempty and compact valued, and upper hemicontinuous.

Steps 2 and 3: Forward induction and the infinite-horizon case. These two steps are the same as those in Appendix A.2, except the corresponding notation need to be changed to be consistent with the perfect information environment whenever necessary.

Remark 4. Theorem 2 remains true if the state transitions either are atomless or have the support inside a fixed finite set irrespective of the history at a particular stage. In the backward induction step, at the stage t that Nature is active and concentrates inside a fixed finite set $\{s_{t1}, \dots, s_{tK}\}$, we have

$$P_t(h_{t-1}, x_t) = \sum_{s_{tk} \in \{s_{t1}, \dots, s_{tK}\}} Q_{t+1}(h_{t-1}, x_t, s_{tk}) f_{t0}\big(\{s_{tk}\} | h_{t-1}\big),$$

where $A_t(h_{t-1}) = \{x_t\}$. Note that P_t is also nonempty and compact valued, and upper hemicontinuous. The proof is the same in other cases. Similarly, Theorem 4 still holds if the state transitions either satisfy the ARM condition or have the support inside a fixed finite set irrespective of the history at a particular stage.

A.4 *Proof of Proposition 1*

The proof is essentially a combination of the proofs in Appendices A.2 and A.3. That is, when there is only one active player, we refer to the argument in Appendix A.3. When there is more than one active player or Nature is the only player that moves, we modify the argument in Appendix A.2.

Step 1: Backward induction. For any $t \geq 1$, suppose that the correspondence Q_{t+1} from H_t to $\mathbb{R}^n$ is bounded, measurable, nonempty and compact valued, and upper hemicontinuous on X^t.

If $N_t = 1$, then $S_t = \{\acute{s}_t\}$. Thus, $P_t(h_{t-1}, x_t) = Q_{t+1}(h_{t-1}, x_t, \acute{s}_t)$, which is nonempty and compact valued, and upper hemicontinuous. Then define the correspondence Φ_t from H_{t-1} to $\mathbb{R}^n \times \mathcal{M}(X_t) \times \triangle(X_t)$ as $(v, \alpha, \mu) \in \Phi_t(h_{t-1})$ if we have the following relationships:

- We have $v = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj})$ such that $p_t(h_{t-1}, \cdot)$ is a measurable selection of $P_t(h_{t-1}, \cdot)$.
- We have that $x^*_{tj} \in A_{tj}(h_{t-1})$ is a maximization point of player j given the payoff function $p_{tj}(h_{t-1}, A_{t(-j)}(h_{t-1}), \cdot)$ and the action space $A_{tj}(h_{t-1})$, $\alpha_i = \delta_{A_{ti}(h_{t-1})}$ for $i \neq j$ and $\alpha_j = \delta_{x^*_{tj}}$.
- We have $\mu = p_t(h_{t-1}, A_{t(-j)}(h_{t-1}), x^*_{tj}) \circ \alpha$.

As discussed in Appendix A.3, Φ_t is nonempty and compact valued, and upper hemicontinuous.

When $N_t = 0$, for any $h_{t-1} \in H_{t-1}$ and $x_t \in A_t(h_{t-1})$,

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t | h_{t-1}, x_t).$$

Let $\operatorname{co} Q_{t+1}(h_{t-1}, x_t, s_t)$ be the convex hull of $Q_{t+1}(h_{t-1}, x_t, s_t)$. Because Q_{t+1} is bounded, nonempty and compact valued, $\operatorname{co} Q_{t+1}$ is bounded, nonempty, convex and compact valued. By Lemma 2(viii), $\operatorname{co} Q_{t+1}$ is upper hemicontinuous.

Note that $f_{t0}(\cdot|h_{t-1}, x_t)$ is atomless and Q_{t+1} is nonempty and compact valued. We have

$$P_t(h_{t-1}, x_t) = \int_{A_{t0}(h_{t-1}, x_t)} \operatorname{co} Q_{t+1}(h_{t-1}, x_t, s_t) f_{t0}(ds_t|h_{t-1}, x_t).$$

By Lemma 7, P_t is bounded, nonempty, convex and compact valued, and upper hemicontinuous. Then by Lemma 10, one can conclude that Φ_t is bounded, nonempty and compact valued, and upper hemicontinuous.

Steps 2 and 3: Forward induction and the infinite-horizon case. These two steps are the same as that in Appendix A.2. The only change is to modify the notations correspondingly.

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